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非线性长波方程组和 Benjamin 方程的新精确孤波解*

套格图桑 斯仁道尔吉
(内蒙古师范大学数学学院,呼和浩特 010022)
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在齐次平衡法、双曲正切函数法和辅助方程法的基础上引入一个新的辅助方程,并借助符号计算系统 Mathematica 来构造了非线性长波方程组和 Benjamin 方程的新精确孤波解,这种方法也可用于寻找其他非线性发展方程的新的孤波解.

关键词:新的辅助方程,非线性长波方程组,Benjamin 方程,孤波解
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1. 引 言

在孤立子理论当中构造非线性发展方程(组)的精确解为占据重要地位.目前已经发现了许多直接方法,比如双曲正切函数法[1],齐次平衡法[2],辅助方程法[3],Jacobi 椭圆函数展开法[4],扩充的双曲正切函数法[5],修正的双曲正切函数法[6]等.我们从这些方法中发现适当选取非线性发展方程的形式解和辅助方程,能够得到非线性发展方程的新精确解.本文在文献[1]—[9]的基础上引入一个新辅助方程,并选取非线性发展方程的一种新形式解,借助计算机代数系统 Mathematica 来构造了非线性长波方程组和 Benjamin 方程的新的精确孤波解,本文给出的方法构造非线性发展方程(组)的精确孤波解方面具有普遍意义.

2. 新辅助方程及其应用步骤

假定给定的非线性发展方程(1+1 维非线性发展方程为例)

$$H(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots) = 0, \tag{1}$$

具有行波解 $u(x,t) = u(\xi)$, $\xi = x + \omega t$,并将其代入方程(1)后得到如下常微分方程:

$$G(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \cdots) = 0. \tag{2}$$

则可以假设方程(2)的解为如下形式[10]:

$$u(\xi) = f_0 + \Big(\sum_{i=1}^{m} f_i z^{-i}(\xi) + \sum_{j=1}^{m} g_j z^j(\xi)\Big) \times (c + az^2(\xi))^{-1}, \tag{3}$$

其中 $f_i(i=0,1,2,\cdots,m)$;$g_j(j=1,2,\cdots,m)$为待定常数,m 为由领头项分析法[11]确定的自然数.在双曲正切函数法中取 $\tanh(\xi) = z(\xi)$,$z(\xi)$是新辅助方程

$$\left(\frac{dz(\xi)}{d\xi}\right)^2 = az^2(\xi) + bz(\xi) + c \tag{4}$$

的解.我们很容易得到一阶常微分方程(4)的如下解:

$$z(\xi) = \frac{1}{a}\left(b - 2\sqrt{ac}\coth\frac{\sqrt{a}}{2}\xi\right)\sinh^2\frac{\sqrt{a}}{2}\xi, (a > 0, c > 0); \tag{5}$$

$$z(\xi) = -\frac{1}{a}\left(b - 2\sqrt{ac}\tanh\frac{\sqrt{a}}{2}\xi\right)\cosh^2\frac{\sqrt{a}}{2}\xi, (a > 0, c > 0); \tag{6}$$

$$z(\xi) = -\frac{1}{a}\left(b + 2\sqrt{-ac}\tan\frac{\sqrt{-a}}{2}\xi\right)\cos^2\frac{\sqrt{-a}}{2}\xi, (a < 0, c > 0); \tag{7}$$

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$$z(\xi)=-\frac{1}{a}\left(b-2\sqrt{-ac}\cot\frac{\sqrt{-a}}{2}\xi\right)\sin^2\frac{\sqrt{-a}}{2}\xi,(a<0,c>0);\tag{8}$$

$$z(\xi)=\frac{1}{4a}\left[-2b-(1+b^2-4ac)\cosh\sqrt{a}\xi+(1-b^2+4ac)\sinh\sqrt{a}\xi\right],(a>0);\tag{9}$$

$$z(\xi)=-\frac{b}{2a}+\frac{b^2-4ac+4a^2(\cosh\sqrt{a}\xi+\sinh\sqrt{a}\xi)^2}{8a^2(\cosh\sqrt{a}\xi+\sinh\sqrt{a}\xi)},(a>0);\tag{10}$$

$$z(\xi)=\frac{b\xi^2}{4}-\frac{c}{b},(a=0);\tag{11}$$

$$z(\xi)=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\sin\sqrt{-a}\xi,(a<0,b^2-4ac>0);\tag{12}$$

$$z(\xi)=\frac{b-2\sqrt{ac}(\tanh\sqrt{a}\xi\pm \mathrm{i}\,\mathrm{sech}\sqrt{a}\xi)}{-a+a(\tanh\sqrt{a}\xi\pm \mathrm{i}\,\mathrm{sech}\sqrt{a}\xi)^2},(c>0,a>0);\tag{13}$$

$$z(\xi)=\frac{b-2\sqrt{ac}(\coth\sqrt{a}\xi\pm \mathrm{csch}\sqrt{a}\xi)}{-a+a(\coth\sqrt{a}\xi\pm \mathrm{csch}\sqrt{a}\xi)^2},(c>0,a>0);\tag{14}$$

$$z(\xi)=-\frac{b+\dfrac{2\sqrt{c}\left(\sqrt{a(P^2+R^2)}-\sqrt{a}P\cosh\sqrt{a}\xi\right)}{R+P\sinh\sqrt{a}\xi}}{a-\dfrac{\left(\sqrt{a(P^2+R^2)}-\sqrt{a}P\cosh\sqrt{a}\xi\right)^2}{(R+P\sinh\sqrt{a}\xi)^2}},(c>0,a>0);\tag{15}$$

$$z(\xi)=\frac{-b+\dfrac{2\sqrt{c}\left(\sqrt{a(-P^2+R^2)}+\sqrt{a}P\sinh\sqrt{a}\xi\right)}{R+P\cosh\sqrt{a}\xi}}{a-\dfrac{\left(\sqrt{a(-P^2+R^2)}+\sqrt{a}P\sinh\sqrt{a}\xi\right)^2}{(R+P\cosh\sqrt{a}\xi)^2}},(c>0,a>0);\tag{16}$$

其中 $c>0,a>0,-P^2+R^2>0,P,R$ 不全为零的任意常数.

$$z(\xi)=\frac{-b+\dfrac{4a\sqrt{c}\cosh\sqrt{a}\xi}{2\sqrt{a}\sinh\sqrt{a}\xi\pm 2\mathrm{i}\sqrt{a}}}{a-\dfrac{4a^2\cosh^2\sqrt{a}\xi}{(2\sqrt{a}\sinh\sqrt{a}\xi\pm 2\mathrm{i}\sqrt{a})^2}},(c>0,a>0);\tag{17}$$

$$z(\xi)=\frac{-b+\dfrac{4a\sqrt{c}\sinh\sqrt{a}\xi}{2\sqrt{a}\cosh\sqrt{a}\xi\pm 2\sqrt{a}}}{a-\dfrac{4a^2\sinh^2\sqrt{a}\xi}{(2\sqrt{a}\cosh\sqrt{a}\xi\pm 2\sqrt{a})^2}},(c>0,a>0);\tag{18}$$

$$z(\xi)=-\frac{b+2\sqrt{-ac}(\tan\sqrt{-a}\xi\pm\sec\sqrt{-a}\xi)}{a[1+(\tan\sqrt{-a}\xi\pm\sec\sqrt{-a}\xi)^2]},(c>0,a<0);\tag{19}$$

$$z(\xi)=\frac{-b+2\sqrt{-ac}(\cot\sqrt{-a}\xi\pm\csc\sqrt{-a}\xi)}{a[1+(\cot\sqrt{-a}\xi\pm\csc\sqrt{-a}\xi)^2]},(c>0,a<0);\tag{20}$$

$$z(\xi)=-\frac{b+\dfrac{4a\sqrt{c}\cos\sqrt{-a}\xi}{2\sqrt{-a}\sin\sqrt{-a}\xi\pm 2\sqrt{-a}}}{a-\dfrac{4a^2\cos^2\sqrt{-a}\xi}{(2\sqrt{-a}\sin\sqrt{-a}\xi\pm 2\sqrt{-a})^2}};\tag{21}$$

$$z(\xi)=\frac{-b+\dfrac{4a\sqrt{c}\sin\sqrt{-a}\xi}{2\sqrt{-a}\cos\sqrt{-a}\xi\pm 2\sqrt{-a}}}{a-\dfrac{4a^2\sin^2\sqrt{-a}\xi}{(2\sqrt{-a}\cos\sqrt{-a}\xi\pm 2\sqrt{-a})^2}};\tag{22}$$

$$z(\xi)=\frac{-b+\dfrac{\sqrt{c}\left[2\sqrt{-a}P\cos\sqrt{-a}\xi\mp2\sqrt{a\left(-P^2+R^2\right)}\right]}{R+P\sin\sqrt{-a}\xi}}{a-\dfrac{\left[-2\sqrt{-a}P\cos\sqrt{-a}\xi\pm2\sqrt{a\left(-P^2+R^2\right)}\right]^2}{4\left(R+P\sin\sqrt{-a}\xi\right)^2}}; \tag{23}$$

$$z(\xi)=-\frac{b+\dfrac{\sqrt{c}\left[2\sqrt{-a}P\sin\sqrt{-a}\xi\pm2\sqrt{a\left(-P^2+R^2\right)}\right]}{R+P\cos\sqrt{-a}\xi}}{a-\dfrac{\left[2\sqrt{-a}P\sin\sqrt{-a}\xi\pm2\sqrt{a\left(-P^2+R^2\right)}\right]^2}{4\left(R+P\cos\sqrt{-a}\xi\right)^2}}; \tag{24}$$

其中 $c>0, a<0, -P^2+R^2<0, P, R$ 不全为零的任意常数.

将(3),(4)式代入(2)式并令 $z(\xi)$ 的各次幂的系数为零后得到一个 $f_i\,(i=0,1,2,\cdots,m)$;$g_j\,(j=1,2,\cdots,m)$,a,b,c,ω 为未知量的非线性代数方程组,利用符号计算系统 Mathematica 或 Maple 求出该方程组的解,再把该非线性代数方程组的每一组解同(5)—(24)式一起代回到(3)式后得到非线性发展方程(组)的精确孤波解.

3.新的精确孤波解

例 1 非线性长波方程组[12—15]

$$u_t+\eta_x+\frac{1}{2}\left(u^2\right)_x=0, \tag{25}$$

$$\eta_t+\left(u\eta+u+u_{xx}\right)_x=0. \tag{26}$$

把 $u(x,t)=u(\xi),\eta(x,t)=\eta(\xi),(\xi=x+\omega t)$ 代入(25),(26)式并对 ξ 积分一次后(积分常数取为零)得到如下常微分方程组:

$$\eta(\xi)=-\omega u(\xi)-\frac{1}{2}u^2(\xi), \tag{27}$$

$$\omega\eta(\xi)+\left(1+\eta(\xi)\right)u(\xi)+u''(\xi)=0. \tag{28}$$

把(27)式代入(28)式经整理后得到如下常微分方程:

$$\left(\omega^2-1\right)u(\xi)+\frac{3}{2}\omega u^2(\xi)+\frac{1}{2}u^3(\xi)-u''(\xi)=0. \tag{29}$$

由领头项分析法可得平衡常数 $m=1$,则取方程(29)的解为如下形式:

$$u(\xi)=f_0+\frac{f_1}{z(\xi)\left(c+az^2(\xi)\right)}+\frac{g_1z(\xi)}{c+az^2(\xi)}, \tag{30}$$

其中 g_1,f_0,f_1 为待定常数,把(4)式和(30)式一起代入(29)式并令 $z(\xi)^j\,(j=0,1,2,3,\cdots,9)$ 的系数为零后得到如下非线性代数方程组:

$$\begin{aligned}
&-4c^3f_1+f_1^3=0,\\
&cf_1\left(-bc+f_1\left(\omega+f_0\right)\right)=0,\\
&f_1\left[c^2\left(-2-14a+2\omega^2+6\omega f_0+3f_0^2\right)\right.\\
&\left.+3f_1g_1\right]=0,\\
&af_1\left[-8bc+\left(3\omega+3f_0\right)f_1\right]+c\left[2c^2\left(-1+\omega^2\right)f_0\right.\\
&\left.+c^2\left(3\omega+f_0\right)f_0^2+\left(-bc+6\omega f_1+6f_0f_1\right)g_1\right]=0,\\
&-28a^2cf_1+2ac\left[\left(-2+2\omega^2\right)f_1+\left(6\omega+3f_0\right)f_0f_1\right.\\
&\left.+5cg_1\right]+g_1\left[-2c^2+2c^2\omega^2+c^2\left(6\omega+3f_0\right)f_0\right.\\
&\left.+3f_1g_1\right]=0,\\
&-7a^2bf_1+c\left(\omega+f_0\right)g_1^2+a\left[2c^2\left(-1+\omega^2\right)f_0\right.\\
&\left.+c^2\left(3\omega+f_0\right)f_0^2+\left(4bc+2\omega f_1\right)g_1+2f_0f_1g_1\right]=0,\\
&-18a^3f_1+2ac\left(-2+2\omega^2+6\omega f_0+3f_0^2\right)g_1+g_1^3\\
&+a^2\left[\left(-2+2\omega^2\right)f_1+\left(6\omega+3f_0\right)f_0f_1+8cg_1\right]=0,\\
&a\left\{\left(\omega+f_0\right)g_1^2+a\left[\left(-2c+2c\omega^2\right)f_0\right.\right.\\
&\left.\left.+c\left(3\omega+f_0\right)f_0^2-bg_1\right]\right\}=0,\\
&a^2\left(-2-2a+2\omega^2+6\omega f_0+3f_0^2\right)g_1=0,\\
&a^3f_0\left(-2+2\omega^2+3\omega f_0+f_0^2\right)=0.
\end{aligned}$$

用符号计算系统 Mathematica 求出该方程组的如下解:

$$b=-2\sqrt{c},\quad f_1=\mp2c\sqrt{c},\quad g_1=0,\quad \omega=\pm1,\quad a=0,\quad f_0=0; \tag{31}$$

$$b=2\sqrt{c},\quad f_1=\mp2c\sqrt{c},\quad g_1=0,\quad \omega=\pm1,\quad a=0,\quad f_0=0; \tag{32}$$

$$b=-2\sqrt{c(1+a)},$$
$$f_1=\mp 2c\sqrt{c},$$
$$g_1=\mp 2a\sqrt{c},$$
$$\omega=\pm\sqrt{1+a},$$
$$f_0=0;\qquad(33)$$
$$b=2\sqrt{c(1+a)},$$
$$f_1=\mp 2c\sqrt{c},$$
$$g_1=\mp 2a\sqrt{c},$$
$$\omega=\mp\sqrt{1+a},$$
$$f_0=0;\qquad(34)$$
$$b=\mp\frac{(12+f_0^2+f_0\sqrt{16+f_0^2})\Delta}{2\sqrt{2}(18+f_0^2)},$$
$$f_1=\mp\frac{c(f_0+3\sqrt{16+f_0^2})\Delta}{2\sqrt{2}(18+f_0^2)},$$
$$g_1=\mp\frac{f_0\Delta}{2\sqrt{2}},$$
$$\omega=\frac{1}{4}(-3f_0+\sqrt{16+f_0^2}),$$
$$a=\frac{1}{8}f_0(-f_0+3\sqrt{16+f_0^2});\qquad(35)$$
$$b=\pm\frac{(12+f_0^2-f_0\sqrt{16+f_0^2})\Omega}{2\sqrt{2}(18+f_0^2)},$$
$$f_1=\pm\frac{c(f_0-3\sqrt{16+f_0^2})\Omega}{2\sqrt{2}(18+f_0^2)},$$
$$g_1=\pm\frac{f_0\Omega}{2\sqrt{2}},$$
$$\omega=\frac{1}{4}(-3f_0-\sqrt{16+f_0^2}),$$
$$a=-\frac{1}{8}f_0(f_0+3\sqrt{16+f_0^2}).\qquad(36)$$

其中

$$\Delta=\sqrt{c(72+5f_0^2-3f_0\sqrt{16+f_0^2})}>0,$$
$$\Omega=\sqrt{c(72+5f_0^2+3f_0\sqrt{16+f_0^2})}>0.$$

把(33)—(36) 式分别与(5)式一起代入(30)和(27)式后得到方程组(25),(26)的如下新的精确孤波解：

$$u_1^{\pm}(x,t)=\pm\frac{a\,\mathrm{csch}\frac{\sqrt{a}}{2}\xi}{\sqrt{a}\cosh\frac{\sqrt{a}}{2}\xi+\sqrt{1+a}\sinh\frac{\sqrt{a}}{a}\xi},$$
$$\eta_1^{\pm}(x,t)=-\omega u_1^{\pm}(x,t)-\frac{1}{2}((u_1^{\pm}(x,t))^2,$$

其中

$$\xi=x\pm\sqrt{1+a}t,a>0;$$
$$u_2^{\pm}(x,t)=\pm\frac{a\,\mathrm{csch}\frac{\sqrt{a}}{2}\xi}{\sqrt{a}\cosh\frac{\sqrt{a}}{2}\xi-\sqrt{1+a}\sinh\frac{\sqrt{a}}{2}\xi},$$
$$\eta_2^{\pm}(x,t)=-\omega u_2^{\pm}(x,t)-\frac{1}{2}(u_2^{\pm}(x,t))^2,$$

其中 $\xi=x\mp\sqrt{1+a}t,a>0;$

$$u_3^{\pm}(x,t)=\frac{1}{4}\left\{4f_0\mp\frac{\sqrt{2}f_0\Delta\left(b-2\sqrt{ac}\coth\frac{\sqrt{a}}{2}\xi\right)\mathrm{csch}^2\frac{\sqrt{a}}{2}\xi}{b^2+ac-4b\sqrt{ac}\coth\frac{\sqrt{a}}{2}\xi+2ac\coth^2\frac{\sqrt{a}}{2}\xi+ac\coth^4\frac{\sqrt{a}}{2}\xi}\right.$$
$$\left.\mp\frac{\sqrt{2}ac\Delta(f_0+3\sqrt{16+f_0^2})\mathrm{csch}^2\frac{\sqrt{a}}{2}\xi}{(18+f_0^2)\left(-b+2\sqrt{ac}\coth\frac{\sqrt{a}}{2}\xi\right)\left[c+\frac{1}{a}\left(b-2\sqrt{ac}\coth\frac{\sqrt{a}}{2}\xi\right)^2\sinh^4\frac{\sqrt{a}}{2}\xi\right]}\right\},$$
$$\eta_3^{\pm}(x,t)=-\omega u_3^{\pm}(x,t)-\frac{1}{2}(u_3^{\pm}(x,t))^2,$$

其中

$$\xi=x+\omega t,\omega=\frac{1}{4}(-3f_0+\sqrt{16+f_0^2}),a=\frac{1}{8}f_0(-f_0+3\sqrt{16+f_0^2}),b=\mp\frac{(12+f_0^2+f_0\sqrt{16+f_0^2})\Delta}{2\sqrt{2}(18+f_0^2)},$$

$a>0$,f_0 为任意不为零的常数；

$$u_4^{\pm}(x,t)=\frac{1}{4}\left\{4f_0\pm\frac{\sqrt{2}f_0\Omega\left(b-2\sqrt{ac}\coth\frac{\sqrt{a}}{2}\xi\right)\mathrm{csch}^2\frac{\sqrt{a}}{2}\xi}{b^2+ac-4b\sqrt{ac}\coth\frac{\sqrt{a}}{2}\xi+2ac\coth^2\frac{\sqrt{a}}{2}\xi+ac\coth^4\frac{\sqrt{a}}{2}\xi}\right.$$

$$\mp \frac{\sqrt{2}ac\Omega\left(f_0-3\sqrt{16+f_0^2}\right)\operatorname{csch}^2\frac{\sqrt{a}}{2}\xi}{\left(18+f_0^2\right)\left(-b+2\sqrt{ac}\coth\frac{\sqrt{a}}{2}\xi\right)\left[c+\frac{1}{a}\left(b-2\sqrt{ac}\coth\frac{\sqrt{a}}{2}\xi\right)^2\sinh^4\frac{\sqrt{a}}{2}\xi\right]}\Bigg\},$$

$$\eta_4^{\pm}(x,t)=-\omega u_4^{\pm}(x,t)-\frac{1}{2}\left(u_4^{\pm}(x,t)\right)^2,$$

其中

$$\xi=x+\omega t,\omega=\frac{1}{4}\left(-3f_0-\sqrt{16+f_0^2}\right),a=-\frac{1}{8}f_0\left(f_0+3\sqrt{16+f_0^2}\right),b=\pm\frac{\left(12+f_0^2-f_0\sqrt{16+f_0^2}\right)\Omega}{2\sqrt{2}\left(18+f_0^2\right)},$$

$a>0$，f_0 为任意不为零的常数.

例 2 Benjamin 方程[16,17]

$$u_{tt}+q\left(u^2\right)_{xx}+ru_{xxxx}=0. \tag{37}$$

把 $u(x,t)=u(\xi)$，$(\xi=x+\omega t)$代入(37)式并对 ξ 积分两次后(积分常数取为零)得到如下常微分方程(只考虑 $r=1$ 的情况)：

$$\omega^2u+qu^2+ru''=0. \tag{38}$$

由领头项分析法可得平衡常数 $m=2$，则取方程(38)的解为如下形式：

$$u(\xi)=f_0+\frac{f_2}{z^2(\xi)\left(c+az^2(\xi)\right)}+\frac{f_1}{z(\xi)\left(c+az^2(\xi)\right)}+\frac{g_1z(\xi)}{c+az^2(\xi)}+\frac{g_2z^2(\xi)}{c+az^2(\xi)}, \tag{39}$$

其中 g_1，g_2，f_0，f_1，f_2 为待定常数，把(4)和(39)式一起代入(38)式并令 $z(\xi)^j$ $(j=0,1,2,3,\cdots,10)$的系数为零后得到如下非线性代数方程组：

$$cf_2\left(6c^2+qf_2\right)=0,$$

$$c\left[5bcf_2+2\left(c^2+qf_2\right)f_1\right]=0,$$

$$2af_2\left(22c^2+qf_2\right)+c\left(2c\omega^2f_2+3bcf_1+2qf_1^2+4cqf_0f_2\right)=0,$$

$$a\left(15bcf_2+7c^2f_1+2qf_2f_1\right)+c\left(c\omega^2f_1+2cqf_0f_1+2qf_2g_1\right)=0,$$

$$64a^2cf_2+2a\left(2c\omega^2f_2+4bcf_1+qf_1^2+4cqf_0f_2\right)+c\left[2c^2\left(\omega^2+qf_0\right)f_0+\left(bc+4qf_1\right)g_1+4\left(c^2+qf_2\right)g_2\right]=0,$$

$$2a^2\left(9bf_2+7cf_1\right)+a\left[\left(2c\omega^2+4cqf_0\right)f_1+\left(-5c^2+2qf_2\right)g_1+c\left[c\left(\omega^2+2qf_0\right)g_1+\left(3bc+2qf_1\right)g_2\right]=0,$$

$$32a^3f_2+a^2\left(2\omega^2f_2+21bf_1+4qf_0f_2\right)+2a\left[3c^2\left(\omega^2+qf_0\right)f_0+\left(-6bc+2qf_1\right)g_1+2\left(-c^2+qf_2\right)g_2\right]+2c\left(qg_1^2+c\omega^2g_2+2cqf_0g_2\right)=0,$$

$$9a^3f_1+a^2\left(\omega^2f_1+2qf_1f_0-4cg_1\right)+2cqg_1g_2+a\left[\left(2c\omega^2+4cqf_0\right)g_1+\left(-5bc+2qf_2\right)g_2\right]=0,$$

$$2cqg_2^2+a^2\left[6c\left(\omega^2+qf_0\right)f_0+3bg_1-8cg_2\right]+2a\left(qg_1^2+2c\omega^2g_2+4cqf_0g_2\right)=0,$$

$$ag_1\left[a^2+a\left(\omega^2+2qf_0\right)+2qg_2\right]=0,$$

$$a\left(af_0+g_2\right)\left[a\left(\omega^2+qf_0\right)+qg_2\right]=0.$$

用符号计算系统 Mathematica 求出该方程组的如下解：

$$b=-\frac{2qg_1}{3a},\quad g_2=-af_0,\quad f_2=0,\quad f_1=0,\quad \omega=\mp\mathrm{i}\sqrt{a},\quad c=0; \tag{40}$$

$$b=-\frac{2qg_1}{3a},\quad g_2=-\frac{a\left(a+qf_0\right)}{q},\quad f_2=0,\quad f_1=0,\quad \omega=\mp\sqrt{a},\quad c=0; \tag{41}$$

$$b=0,\quad g_2=0,\quad g_1=0,\quad f_2=-\frac{6c^2}{q},\quad f_1=0,\quad \omega=\mp4\mathrm{i}\sqrt{-a},\quad f_0=-\frac{16a}{q}; \tag{42}$$

$$b=0,\quad g_2=-\frac{6a^2}{q},\quad g_1=0,\quad f_2=0,\quad f_1=0,\quad \omega=\mp 2\sqrt{a},\quad f_0=\frac{2a}{q};\tag{43}$$

$$b=0,\quad g_2=-\frac{3a^2}{q},\quad g_1=-3\mathrm{i}\left|\frac{a}{q}\right|\sqrt{ca},\quad f_2=0,\quad f_1=0,\quad \omega=\mp\sqrt{a},\quad f_0=\frac{2a}{q};\tag{44}$$

$$b=0,\quad g_2=-\frac{3a^2}{q},\quad g_1=3\mathrm{i}\left|\frac{a}{q}\right|\sqrt{ca},\quad f_2=0,\quad f_1=0,\quad \omega=\mp\sqrt{a},\quad f_0=\frac{2a}{q};\tag{45}$$

$$b=0,\quad g_2=-\frac{3a^2}{q},\quad g_1=-3\mathrm{i}\left|\frac{a}{q}\right|\sqrt{ca},\quad f_2=0,\quad f_1=0,\quad \omega=\mp\mathrm{i}\sqrt{a},\quad f_0=\frac{3a}{q};\tag{46}$$

$$b=0,\quad g_2=-\frac{3a^2}{q},\quad g_1=3\mathrm{i}\left|\frac{a}{q}\right|\sqrt{ca},\quad f_2=0,\quad f_1=0,\quad \omega=\mp\mathrm{i}\sqrt{a},\quad f_0=\frac{3a}{q};\tag{47}$$

$$b=0,\quad g_2=-\frac{6a^2}{q},\quad g_1=0,\quad f_2=0,\quad f_1=0,\quad \omega=\mp 2\mathrm{i}\sqrt{a},\quad f_0=\frac{6a}{q};\tag{48}$$

$$b=0,\quad g_2=\frac{6a^2}{q},\quad g_1=0,\quad f_2=-\frac{6c^2}{q},\quad f_1=0,\quad \omega=\mp 2\mathrm{i}\sqrt{-a},\quad f_0=-\frac{10a}{q};\tag{49}$$

$$b=0,\quad g_2=\frac{6a^2}{q},\quad g_1=0,\quad f_2=-\frac{6c^2}{q},\quad f_1=0,\quad \omega=\mp 2\sqrt{-a},\quad f_0=-\frac{6a}{q};\tag{50}$$

$$b=-2\mathrm{i}\sqrt{-ac},\quad g_2=\frac{6a^2}{q},\quad g_1=-6\mathrm{i}\left|\frac{a}{q}\right|\sqrt{-ac},\quad f_2=-\frac{6c^2}{q},\quad f_1=6\mathrm{i}\left|\frac{c}{q}\right|\sqrt{-ac},\quad \omega=\mp\sqrt{-a},\quad f_0=-\frac{6a}{q};\tag{51}$$

$$
\begin{aligned}
b &= 2\mathrm{i}\sqrt{-ac}, \\
g_2 &= \frac{6a^2}{q}, \\
g_1 &= 6\mathrm{i}\left|\frac{a}{q}\right|\sqrt{-ac}, \\
f_2 &= -\frac{6c^2}{q}, \\
f_1 &= -6\mathrm{i}\left|\frac{c}{q}\right|\sqrt{-ac}, \\
\omega &= \mp\sqrt{-a}, \\
f_0 &= -\frac{6a}{q};
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
b &= -2\mathrm{i}\sqrt{-ac}, \\
g_2 &= \frac{6a^2}{q}, \\
g_1 &= -6\mathrm{i}\left|\frac{a}{q}\right|\sqrt{-ac}, \\
f_2 &= -\frac{6c^2}{q}, \\
f_1 &= 6\mathrm{i}\left|\frac{c}{q}\right|\sqrt{-ac}, \\
\omega &= \mp\mathrm{i}\sqrt{-a}, \\
f_0 &= -\frac{7a}{q};
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
b &= 2\mathrm{i}\sqrt{-ac}, \\
g_2 &= \frac{6a^2}{q}, \\
g_1 &= 6\mathrm{i}\left|\frac{a}{q}\right|\sqrt{-ac}, \\
f_2 &= -\frac{6c^2}{q}, \\
f_1 &= -6\mathrm{i}\left|\frac{c}{q}\right|\sqrt{-ac}, \\
\omega &= \mp\mathrm{i}\sqrt{-a}, \\
f_0 &= -\frac{7a}{q}.
\end{aligned} \tag{54}
$$

把(40)—(54)式分别与(10)式一起代入(39)式后得到方程(37)的如下新的精确孤波解：

$$u_{\pm}^{1}(x,t) = \frac{18a^3 g_1(\cosh\sqrt{a}\xi + \sinh\sqrt{a}\xi)}{[qg_1 + 3a^2(\cosh\sqrt{a}\xi + \sinh\sqrt{a}\xi)]^2}, \xi = x \mp \mathrm{i}\sqrt{a}t, (a < 0);$$

$$u_{\pm}^{2}(x,t) = -\frac{a}{q} + \frac{18a^3 g_1(\cosh\sqrt{a}\xi + \sinh\sqrt{a}\xi)}{[qg_1 + 3a^2(\cosh\sqrt{a}\xi + \sinh\sqrt{a}\xi)]^2}, \xi = x \mp \sqrt{a}t, (a > 0);$$

$$u_{\pm}^{3}(x,t) = \frac{16a}{q}\left\{-1 - \frac{6a^2c^2}{[(M\cosh\sqrt{a}\xi + N\sinh\sqrt{a}\xi)(N\cosh\sqrt{a}\xi + M\sinh\sqrt{a}\xi)]^2}\right\},$$
$$\xi = x \mp 4\mathrm{i}\sqrt{-a}t, (a > 0);$$

$$u_{\pm}^{4}(x,t) = -\frac{4a[-4ac + (a^2 + c^2)\cosh2\sqrt{a}\xi + MN\sinh2\sqrt{a}\xi]}{q[M\cosh\sqrt{a}\xi + N\sinh\sqrt{a}\xi]^2}, \xi = x \mp 2\sqrt{a}t, (a > 0);$$

$$u_{\pm}^{5}(x,t) = -\frac{a[-10ac + 6\mathrm{i}N\sqrt{ac}\cosh\sqrt{a}\xi + (a^2 + c^2)\cosh2\sqrt{a}\xi]}{q[M\cosh\sqrt{a}\xi + N\sinh\sqrt{a}\xi]^2}$$
$$+ \frac{a[6\mathrm{i}(|c| + |a|)\sqrt{ac}\sinh\sqrt{a}\xi + MN\sinh2\sqrt{a}\xi]}{q[M\cosh\sqrt{a}\xi + N\sinh\sqrt{a}\xi]^2}, \xi = x \mp \sqrt{a}t, (a > 0);$$

$$u_{\pm}^{6}(x,t) = -\frac{a[-10ac - 6\mathrm{i}N\sqrt{ac}\cosh\sqrt{a}\xi + (a^2 + c^2)\cosh2\sqrt{a}\xi]}{q[M\cosh\sqrt{a}\xi + N\sinh\sqrt{a}\xi]^2}$$
$$+ \frac{a[-6\mathrm{i}(|c| + |a|)\sqrt{ac}\sinh\sqrt{a}\xi + MN\sinh2\sqrt{a}\xi]}{q[M\cosh\sqrt{a}\xi + N\sinh\sqrt{a}\xi]^2}, \xi = x \mp \sqrt{a}t, (a > 0);$$

$$u_{\pm}^{7}(x,t) = \frac{6a\sqrt{c}[2a\sqrt{c} - \mathrm{i}N\sqrt{a}\cosh\sqrt{a}\xi - \mathrm{i}M\sqrt{a}\sinh\sqrt{a}\xi]}{q[M\cosh\sqrt{a}\xi + N\sinh\sqrt{a}\xi]^2}, \xi = x \mp \mathrm{i}\sqrt{a}t, (a < 0);$$

$$u_{\pm}^{8}(x,t) = \frac{6a\sqrt{c}[2a\sqrt{c} + \mathrm{i}N\sqrt{a}\cosh\sqrt{a}\xi + \mathrm{i}M\sqrt{a}\sinh\sqrt{a}\xi]}{q[M\cosh\sqrt{a}\xi + N\sinh\sqrt{a}\xi]^2}, \xi = x \mp \mathrm{i}\sqrt{a}t, (a < 0);$$

$$u_{\pm}^{9}(x,t) = \frac{24a^2c}{q[M\cosh\sqrt{a}\xi + N\sinh\sqrt{a}\xi]^2}, \xi = x \mp 2\mathrm{i}\sqrt{a}t, (a < 0);$$

$$u_{\pm}^{10}(x,t)=\frac{-4a[4ac+(a^2+c^2)\cosh 2\sqrt{a}\xi+MN\sinh 2\sqrt{a}\xi]\Theta(\xi)}{q[-c+a\cosh^2\sqrt{a}\xi+a\sinh^2\sqrt{a}\xi+a\sinh 2\sqrt{a}\xi]^2},\xi=x\mp 2\mathrm{i}\sqrt{-a}t,(a>0);$$

$$u_{\pm}^{11}(x,t)=\frac{-24a^2c}{q[M\cosh\sqrt{a}\xi+N\sinh\sqrt{a}\xi]^2},\xi=x\mp 2\sqrt{-a}t,(a<0);$$

$$u_{\pm}^{12}(x,t)=\frac{12a^2\sqrt{c}[4a\sqrt{c}+\mathrm{i}(a+8c)\sqrt{-a}\cosh\sqrt{a}\xi+\mathrm{i}(a-8c)\sqrt{-a}\sinh\sqrt{a}\xi]\Theta(\xi)}{q(\cosh\sqrt{a}\xi+\sinh\sqrt{a}\xi)[2\mathrm{i}\sqrt{-ac}+a\cosh\sqrt{a}\xi+a\sinh\sqrt{a}\xi]^2\Upsilon(\xi)},$$

$$\xi=x\mp\sqrt{-a}t,(a<0);$$

$$u_{\pm}^{13}(x,t)=\frac{12a^2\sqrt{c}[4a\sqrt{c}-\mathrm{i}(a+8c)\sqrt{-a}\cosh\sqrt{a}\xi-\mathrm{i}(a-8c)\sqrt{-a}\sinh\sqrt{a}\xi]\Theta(\xi)}{q(\cosh\sqrt{a}\xi+\sinh\sqrt{a}\xi)[-2\mathrm{i}\sqrt{-ac}+a\cosh\sqrt{a}\xi+a\sinh\sqrt{a}\xi]^2\Psi(\xi)},$$

$$\xi=x\mp\sqrt{-a}t,(a<0);$$

$$u_{\pm}^{14}(x,t)=\frac{-a^2[-20ac-4\mathrm{i}(a+12c)\sqrt{-ac}\cosh\sqrt{a}\xi+(a^2+32c^2)\cosh 2\sqrt{a}\xi]\Theta(\xi)}{q(\cosh\sqrt{a}\xi+\sinh\sqrt{a}\xi)[2\mathrm{i}\sqrt{-ac}+a\cosh\sqrt{a}\xi+a\sinh\sqrt{a}\xi]^2\Upsilon(\xi)}$$
$$+\frac{-a^2[(48\mathrm{i}|c|+4\mathrm{i}|a|)\sqrt{-ac}\sinh\sqrt{a}\xi+(a^2-32c^2)\sinh 2\sqrt{a}\xi]\Theta(\xi)}{q(\cosh\sqrt{a}\xi+\sinh\sqrt{a}\xi)[2\mathrm{i}\sqrt{-ac}+a\cosh\sqrt{a}\xi+a\sinh\sqrt{a}\xi]^2\Upsilon(\xi)},$$

$$\xi=x\mp\mathrm{i}\sqrt{-a}t,(a>0,c>0);$$

$$u_{\pm}^{15}(x,t)=\frac{-a^2[-20ac+4\mathrm{i}(a+12c)\sqrt{-ac}\cosh\sqrt{a}\xi+(a^2+32c^2)\cosh 2\sqrt{a}\xi]\Theta(\xi)}{q(\cosh\sqrt{a}\xi+\sinh\sqrt{a}\xi)[-2\mathrm{i}\sqrt{-ac}+a\cosh\sqrt{a}\xi+a\sinh\sqrt{a}\xi]^2\Psi(\xi)}$$
$$+\frac{-a^2[(4\mathrm{i}a-48\mathrm{i}|c|)\sqrt{-ac}\sinh\sqrt{a}\xi+(a^2-32c^2)\sinh 2\sqrt{a}\xi]\Theta(\xi)}{q(\cosh\sqrt{a}\xi+\sinh\sqrt{a}\xi)[-2\mathrm{i}\sqrt{-ac}+a\cosh\sqrt{a}\xi+a\sinh\sqrt{a}\xi]^2\Psi(\xi)},$$

$$\xi=x\mp\mathrm{i}\sqrt{-a}t,(a>0,c>0).$$

$$M=a+c,N=a-c,\Theta(\xi)=\cosh 2\sqrt{a}\xi+\sinh 2\sqrt{a}\xi,$$

$$\gamma(\xi)=4\mathrm{i}\sqrt{-ac}+(4+8c)\cosh\sqrt{a}\xi+(a-8c)\sinh\sqrt{a}\xi,$$

$$\Psi(\xi)=-4\mathrm{i}\sqrt{-ac}+(a+8c)\cosh\sqrt{a}\xi+(a-8c)\sinh\sqrt{a}\xi,(a>0,c>0).$$

4. 结　论

本文引入一个新的辅助方程，并把非线性发展方程的解取为一个新形式，借助计算机代数系统 Mathematica 来构造了非线性长波方程组的形如 $\begin{cases}u_1^{\pm}(x,t)\\ \eta_1^{\pm}(x,t)\end{cases}$—$\begin{cases}u_4^{\pm}(x,t)\\ \eta_4^{\pm}(x,t)\end{cases}$ 和 Benjamin 方程的形如 $u_{\pm}^{1}(x,t)-u_{\pm}^{15}(x,t)$ 的新的精确孤波解.(31)和(32)式只得到非线性长波方程组的有理解.辅助方程(4)和形式解(3)构造的非线性长波方程组和 Benjamin 方程的其余解，限于篇幅未能列出.本文给出的形式解(3)结合其他辅助方程，也得到许多非线性发展方程(组)的新精确孤波解.

[1] Parkes E J, Duffy B R 1996 *Comput. Phys. Commun.* **98** 288

[2] Wang M L 1995 *Phys. Lett.* A **199** 169

[3] Sirendaoreji, Sun J 2003 *Phys. Lett.* A **309** 387

[4] Liu S K, Fu Z T, Liu S D *et al* 2002 *Acta Phys. Sin.* **51** 10 (in Chinese) [刘式适、付遵涛、刘式达等 2002 物理学报 **51** 10]

[5] Fan E G 2000 *Phys. Lett.* A **277** 212

[6] Elwakil S A, El-labany S K, Zaharan M A *et al* 2002 *Phys. Lett.* A **299** 179

[7] Zheng X D, Chen y, Zhang H Q 2003 *Commun. Theor. Phys.* **39** 647

[8] Chen Y, Yan Z Y, Li B *et al* 2003 *Chin. Phys.* **12** 1

[9] Lü Z S, Zhang H Q 2003 *Commun. Theor. Phys.* **39** 405

[10] Xie F D, Gao X S 2004 *Commun. Theor. Phys.* **41** 353

[11] Malfliet W 1992 *Am. J. Phys.* **60** 650

[12] Zhang J F, Guo G P, Wu F M 2002 *Chin. Phys.* **11** 533

[13] Zhou Z J，Li Z B 2003 *Acta Phys. Sin.* **52** 262(in Chinese)[周振江、李志斌 2003 物理学报 **52** 262]

[14] Narenmandula 2002 *Acta Phys. Sin.* **51** 1671(in Chinese)[那仁满都拉 2002 物理学报 **51** 1671]

[15] Guo F K，Zhang Y F 2002 *Acta Phys. Sin.* **51** 951(in Chinese)[郭福奎、张玉峰 2002 物理学报 **51** 951]

[16] Xu G Q，Li Z B 2003 *Acta Phys. Sin.* **52** 1848(in Chinese)[徐桂琼、李志斌 2003 物理学报 **52** 1848]

[17] Sirendaoreji 2003 *Journal of Inner Mongolia Normal University* **32** 1(in Chinese)[斯仁道尔吉 2003 内蒙古师范大学学报(自然科学版) **32** 1]

New type of exact solitary wave solutions for dispersive long-wave equation and Benjamin equation*

Taogetusang Sirendaoreji

(*School of Mathematical Science*，*Inner Mongolia Normal University*，*Huhhot* 010022，*China*)

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Abstract

On the basis of homogeneous balance method，tanh-function method and auxiliary equation method，new types of exact solitary wave solutions of the dispersive long-wave equation and Benjamin equation are constructed by using a new auxiliary equation and with the aid of computer algebraic system Mathematica in this paper. This method also can be used to find new solitary wave solutions of other nonlinear evolution equations.

Keywords：new auxiliary equation，dispersive long-wave equation，Benjamin equation，exact solitary wave solution

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