第 55 卷 第 8 期 2006 年 8 月
1000-3290/2006/55(08)/3821-04

物 理 学 报
ACTA PHYSICA SINICA

Vol.55, No.8, August, 2006

相空间中力学系统的 Lie-Mei 对称性

方建会 王 鹏 丁 宁

(中国石油大学(华东)物理科学与技术学院，东营 257061)

(2005 年 10 月 18 日收到；2005 年 11 月 18 日收到修改稿)

研究了相空间中力学系统的一种新对称性——Lie-Mei 对称性及其守恒量. 提出这种新对称性的定义，给出了系统 Lie-Mei 对称性的判据，得到了系统 Lie-Mei 对称性导致的广义 Hojman 守恒量和 Mei 守恒量. 举例说明了结果的应用.

关键词：相空间，力学系统，Lie-Mei 对称性，守恒量
PACC：0320

1. 引　　言

力学系统对称性与守恒量的研究具有重要的理论意义和实际价值. 利用动力学系统的对称性来寻找系统的守恒量是分析力学的一个近代发展方向，近年来备受关注. 力学系统的近代对称性理论主要有 Noether 对称性、Lie 对称性和 Mei 对称性[1—10]. 得到的守恒量主要有 Noether 守恒量、Hojman 守恒量和新型守恒量(也称 Mei 守恒量)[1,11]. 在以往的研究中通常都是由一种对称性寻求一种类型的守恒量. 最近，梅凤翔等[12,13]将上述三种对称性联合，研究了位形空间中力学系统的统一对称性，由统一对称性同时得到了上述的三种守恒量. 吴惠彬[14]研究了位形空间中 Lagrange 系统的 Lie 形式不变性及其守恒量. 本文研究相空间中力学系统的 Lie-Mei 对称性，给出相空间中力学系统 Lie-Mei 对称性的定义和判据，得到系统 Lie-Mei 对称性导致的广义 Hojman 守恒量和 Mei 守恒量.

2. Lie-Mei 对称性的定义和判据

相空间中力学系统的运动微分方程为

$$\dot{q}_s = \frac{\partial H}{\partial p_s},\qquad \dot{p}_s = -\frac{\partial H}{\partial q_s} + Q_s \qquad (s = 1,\cdots,s), \tag{1}$$

式中，$H = H(t,\boldsymbol{q},\boldsymbol{p})$ 是系统的 Hamilton 函数，$Q_s = Q_s(t,\boldsymbol{q},\boldsymbol{p})$ 为非势广义力. 展开方程(1)，并记作

$$\dot{q}_s = g_s(t,\boldsymbol{q},\boldsymbol{p}),\qquad \dot{p}_s = h_s(t,\boldsymbol{q},\boldsymbol{p}). \tag{2}$$

引入相空间的无限小群变换

$$t^* = t + \varepsilon\xi_0(t,\boldsymbol{q},\boldsymbol{p}),\qquad q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t,\boldsymbol{q},\boldsymbol{p}),\qquad p_s^*(t^*) = p_s(t) + \varepsilon\eta_s(t,\boldsymbol{q},\boldsymbol{p}), \tag{3}$$

式中，ε 为无限小参量，ξ_0,ξ_s,η_s 为无限小变换的生成元.

定义 如果力学系统(1)的一个对称性既是 Lie 对称性又是 Mei 对称性，则称该对称性为系统(1)的 Lie-Mei 对称性.

方程(1)的 Lie 对称性确定方程为

$$X^{(1)}\left(\dot{q}_s - \frac{\partial H}{\partial p_s}\right) = 0,\qquad X^{(1)}\left(\dot{p}_s + \frac{\partial H}{\partial q_s} - Q_s\right) = 0, \tag{4}$$

式中，

$$X^{(1)} = X^{(0)} + \left(\frac{\bar{\mathrm{d}}\,\xi_s}{\mathrm{d}t} - \dot{q}_s\frac{\bar{\mathrm{d}}\,\xi_0}{\mathrm{d}t}\right)\frac{\partial}{\partial \dot{q}_s} + \left(\frac{\bar{\mathrm{d}}\,\eta_s}{\mathrm{d}t} - \dot{p}_s\frac{\bar{\mathrm{d}}\,\xi_0}{\mathrm{d}t}\right)\frac{\partial}{\partial \dot{p}_s}, \tag{5}$$

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \eta_s\frac{\partial}{\partial p_s}, \tag{6}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + g_s\frac{\partial}{\partial q_s} + h_s\frac{\partial}{\partial p_s}. \tag{7}$$

方程(4)也可写成下列等价形式：

$$\frac{\bar{\mathrm{d}}\,\xi_s}{\mathrm{d}t} - g_s\frac{\bar{\mathrm{d}}\,\xi_0}{\mathrm{d}t} = X^{(0)}(g_s),$$

$$\frac{\bar{d}\eta_s}{dt}-h_s\frac{\bar{d}\xi_0}{dt}=X^{(0)}(h_s). \tag{8}$$

方程(1)的 Mei 对称性确定方程为

$$\begin{aligned}&\frac{\partial}{\partial p_s}[X^{(0)}(H)]=0,\\&\frac{\partial}{\partial q_s}[X^{(0)}(H)]-X^{(0)}(Q_s)=0.\end{aligned} \tag{9}$$

根据定义,利用(4),(9)式可得以下判据.

判据 对相空间中的力学系统(1),如果无限小变换的生成元 ξ_0,ξ_s,η_s 满足

$$\begin{aligned}&\left\{X^{(1)}\left(\dot{q}_s-\frac{\partial H}{\partial p_s}\right)\right\}^2+\left\{X^{(1)}\left(\dot{p}_s+\frac{\partial H}{\partial q_s}-Q_s\right)\right\}^2\\&+\left\{\frac{\partial}{\partial p_s}[X^{(0)}(H)]\right\}^2+\left\{\frac{\partial}{\partial q_s}[X^{(0)}(H)]-X^{(0)}(Q_s)\right\}^2\\&=0,\end{aligned} \tag{10}$$

则相应的对称性是系统的 Lie-Mei 对称性.

3. Lie-Mei 对称性导致的守恒量

相空间中力学系统的 Lie-Mei 对称性在一定条件下可导致广义 Hojman 守恒量和 Mei 守恒量.

命题 1 对力学系统(1),如果在无限小变换(3)式下,存在函数 $u=u(t,\boldsymbol{q},\boldsymbol{p})$ 满足

$$\frac{\partial g_s}{\partial q_s}+\frac{\partial h_s}{\partial p_s}+\frac{\bar{d}}{dt}\ln u=0, \tag{11}$$

则系统的 Lie-Mei 对称性导致守恒量

$$\begin{aligned}I_H=&\frac{1}{u}\frac{\partial(u\xi_0)}{\partial t}+\frac{1}{u}\frac{\partial(u\xi_s)}{\partial q_s}+\frac{1}{u}\frac{\partial(u\eta_s)}{\partial p_s}-\frac{\bar{d}\xi_0}{dt}\\&=\text{const}.\end{aligned} \tag{12}$$

证明 我们有

$$\begin{aligned}\frac{\bar{d}I_H}{dt}=&\frac{\bar{d}}{dt}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\xi_0\right)+\frac{\bar{d}}{dt}\frac{\partial\xi_0}{\partial t}+\frac{\bar{d}}{dt}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right)\\&+\frac{\bar{d}}{dt}\frac{\partial\xi_s}{\partial q_s}+\frac{\bar{d}}{dt}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial p_s}\eta_s\right)+\frac{\bar{d}}{dt}\frac{\partial\eta_s}{\partial p_s}\\&-\frac{\bar{d}}{dt}\left(\frac{\bar{d}\xi_0}{dt}\right),\end{aligned} \tag{13}$$

$$\frac{\bar{d}}{dt}\frac{\partial\xi_0}{\partial t}=\frac{\partial}{\partial t}\frac{\bar{d}\xi_0}{dt}-\frac{\partial\xi_0}{\partial q_k}\frac{\partial g_k}{\partial t}-\frac{\partial\xi_0}{\partial p_k}\frac{\partial h_k}{\partial t}, \tag{14}$$

$$\frac{\bar{d}}{dt}\frac{\partial\xi_s}{\partial q_s}=\frac{\partial}{\partial q_s}\frac{\bar{d}\xi_s}{dt}-\frac{\partial\xi_s}{\partial q_k}\frac{\partial g_k}{\partial q_s}-\frac{\partial\xi_s}{\partial p_k}\frac{\partial h_k}{\partial q_s}, \tag{15}$$

$$\frac{\bar{d}}{dt}\frac{\partial\eta_s}{\partial p_s}=\frac{\partial}{\partial p_s}\frac{\bar{d}\eta_s}{dt}-\frac{\partial\eta_s}{\partial q_k}\frac{\partial g_k}{\partial p_s}-\frac{\partial\eta_s}{\partial p_k}\frac{\partial h_k}{\partial p_s}. \tag{16}$$

从定义知,力学系统(1)的 Lie-Mei 对称性也一定是 Lie 对称性,将(14)—(16)式代入(13)式并利用(8)式,可得

$$\begin{aligned}\frac{\bar{d}I_H}{dt}=&\frac{\bar{d}}{dt}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\xi_0\right)+\frac{\bar{d}}{dt}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right)+\frac{\bar{d}}{dt}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial p_s}\eta_s\right)\\&+\frac{\partial g_s}{\partial q_s}\frac{\bar{d}\xi_0}{dt}+\frac{\partial h_s}{\partial p_s}\frac{\bar{d}\xi_0}{dt}+\xi_0\frac{\partial^2 g_s}{\partial q_s\partial t}+\xi_0\frac{\partial^2 h_s}{\partial p_s\partial t}\\&+\frac{\partial^2 g_s}{\partial q_s\partial q_k}\xi_k+\frac{\partial^2 g_s}{\partial q_s\partial p_k}\eta_k+\frac{\partial^2 h_s}{\partial p_s\partial q_k}\xi_k+\frac{\partial^2 h_s}{\partial p_s\partial p_k}\eta_k.\end{aligned} \tag{17}$$

将条件(11)式对 t,q_k 和 p_k 求偏导数,并将其代入(17)式,可得

$$\begin{aligned}\frac{\bar{d}I_H}{dt}=&\frac{\bar{d}}{dt}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\xi_0\right)+\frac{\bar{d}}{dt}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right)\\&+\frac{\bar{d}}{dt}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial p_s}\eta_s\right)-\xi_0\frac{\partial}{\partial t}\frac{\bar{d}}{dt}\ln\mu\\&-\xi_k\frac{\partial}{\partial q_k}\frac{\bar{d}}{dt}\ln\mu-\eta_k\frac{\partial}{\partial p_k}\frac{\bar{d}}{dt}\ln\mu-\frac{\bar{d}\xi_0}{dt}\frac{\bar{d}}{dt}\ln\mu\\=&\frac{\partial}{\partial q_k}\ln\mu\left(\frac{\bar{d}\xi_k}{dt}-\xi_0\frac{\partial g_k}{\partial t}-\xi_s\frac{\partial g_k}{\partial q_s}-\eta_s\frac{\partial g_k}{\partial p_s}\right)\\&+\frac{\partial}{\partial p_k}\ln\mu\left(\frac{\bar{d}\eta_k}{dt}-\xi_0\frac{\partial h_k}{\partial t}-\xi_s\frac{\partial h_k}{\partial q_s}-\eta_k\frac{\partial h_k}{\partial p_s}\right)\\&+\frac{\bar{d}\xi_0}{dt}\frac{\partial}{\partial t}\ln\mu-\frac{\bar{d}\xi_0}{dt}\frac{\bar{d}}{dt}\ln\mu.\end{aligned} \tag{18}$$

考虑到(8)式,则有

$$\frac{\bar{d}I_H}{dt}=0. \tag{19}$$

我们称守恒量(12)式为相空间中力学系统的广义 Hojman 守恒量. 当 $\xi_0=0$ 时,守恒量(12)式化为

$$I_H=\frac{1}{\mu}\frac{\partial(u\xi_s)}{\partial q_s}+\frac{1}{u}\frac{\partial(u\eta_s)}{\partial p_s}=\text{const}. \tag{20}$$

这就是文献[15]中给出的相空间中力学系统的 Hojman 守恒量.对 Hamilton 系统,(12)式化为

$$I_H=\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_s}{\partial q_s}+\frac{\partial\eta_s}{\partial p_s}-\frac{\bar{d}\xi_0}{dt}=\text{const}. \tag{21}$$

这正是文献[16]中给出的结果. 当 $\xi_0=0$ 时,(21)式化为

$$I_H=\frac{\partial\xi_s}{\partial q_s}+\frac{\partial\eta_s}{\partial p_s}=\text{const}. \tag{22}$$

这就是文献[1]中给出的 Hamilton 系统的 Hojman 守恒量.

命题 2 对力学系统(1),如果存在函数 $G_M(t,\boldsymbol{q},\boldsymbol{p})$ 使无限小变换的生成元满足

$$\begin{aligned}&X^{(0)}(p_s)\frac{\bar{d}\xi_s}{dt}+\frac{\bar{d}}{dt}[X^{(0)}(p_s)]\xi_s\\&-X^{(0)}[X^{(0)}(H)]-X^{(0)}(H)\frac{\bar{d}\xi_0}{dt}\end{aligned}$$

$$+X^{(0)}(Q_s)(\xi_s-\dot{q}_s\xi_0)+\frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t}=0, \tag{23}$$

则系统的 Lie-Mei 对称性导致 Mei 守恒量

$$I_{\mathrm{M}}=X^{(0)}(p_s)\xi_s-X^{(0)}(H)\xi_0+G_{\mathrm{M}}=\mathrm{const}. \tag{24}$$

证明　我们有

$$\frac{\bar{\mathrm{d}}I_M}{\mathrm{d}t}=X^{(0)}(p_s)\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}[X^{(0)}(p_s)]\xi_s-X^{(0)}(H)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}-\left[\frac{\partial X^{(0)}(H)}{\partial t}+\dot{q}_s\frac{\partial X^{(0)}(H)}{\partial q_s}+\dot{p}_s\frac{\partial X^{(0)}(H)}{\partial p_s}\right]\xi_0+\frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t}. \tag{25}$$

从定义知，力学系统(1)的 Lie-Mei 对称性也一定是 Mei 对称性，因此将 Mei 对称性的确定方程(9)代入(25)式并利用(23)式，可得

$$\frac{\bar{\mathrm{d}}I_M}{\mathrm{d}t}=0. \tag{26}$$

4. 算　例

力学系统的 Hamilton 函数为

$$H=\frac{1}{2}(p_1^2+p_2^2)+q_2, \tag{27}$$

系统受到的非势广义力为 $Q_1=t+p_1+p_2$，$Q_2=0$ 试研究其 Lie-Mei 对称性.

运动微分方程(1)给出

$$\begin{aligned}g_1&=\dot{q}_1=p_1,\\ g_2&=\dot{q}_2=p_2,\\ h_1&=\dot{p}_1=t+p_1+p_2,\\ h_2&=\dot{p}_2=-1.\end{aligned} \tag{28}$$

计算可得

$$X^{(0)}(H)=\eta_1p_1+\eta_2p_2+\xi_2, \tag{29}$$

$$X^{(0)}(Q_1)=\xi_0+\eta_1+\eta_2. \tag{30}$$

取

$$\begin{aligned}\xi_0&=1,\\ \xi_1&=\left(\frac{1}{2}t^2+q_1+q_2-p_1\right)^2,\\ \xi_2&=p_2,\\ \eta_1&=0,\\ \eta_2&=-1,\end{aligned} \tag{31}$$

则 $X^{(0)}(H)=0$，$X^{(0)}(Q_1)=0$，系统 Lie-Mei 对称性的判据方程(10)满足.

由(11)式可得

$$u=\mathrm{e}^{-t}. \tag{32}$$

命题 1 给出

$$I_{\mathrm{H}}=2\left(\frac{1}{2}t^2+q_1+q_2-p_1\right)-1=\mathrm{const}. \tag{33}$$

由方程(23)可得

$$G_{\mathrm{M}}=-t. \tag{34}$$

命题 2 给出

$$I_{\mathrm{M}}=-p_2-t=\mathrm{const}. \tag{35}$$

5. 结　论

本文给出的相空间中力学系统的新对称性——Lie-Mei 对称性，包含了 Lie 对称性和 Mei 对称性，具有更普遍的意义. 以往关于相空间中力学系统对称性与守恒量的研究大都是由一种对称性得到一类守恒量，本文由 Lie-Mei 对称性同时得到了系统的广义 Hojman 守恒量和 Mei 守恒量. 通常的 Hojman 守恒量都是在 $\xi_0=0$ 的特殊无限小变换下得到的，而本文给出的广义 Hojman 守恒量是在 $\xi_0\neq0$ 的一般无限小变换下得到的，具有更一般的意义. 当 $\xi_0=0$ 时，本文的广义 Hojman 守恒量便化为 Hojman 守恒量.

[1] Mei F X 2004 *Symmetries and Conserved Quantities of Constrained Mechanical Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese)[梅凤翔 2004 约束力学系统的对称性与守恒量(北京 北京理工大学出版社)]

[2] Lutzky M 1979 *J. Phys. A: Math. Gen.* **12** 973

[3] Fu J L, Chen L Q 2004 *Mech. Res. Commun.* **31** 9

[4] Guo Y X, Jiang L Y, Yu Y 2001 *Chin. Phys.* **10** 181

[5] Zhang H B 2002 *Chin. Phys.* **11** 1

[6] Luo S K 2003 *Acta Phys. Sin.* **52** 2941 (in Chinese)[罗绍凯 2003 物理学报 **52** 2941]

[7] Fang J H 2004 *Commun. Theor. Phys.* **41** 349

[8] Lou Z M 2004 *Acta Phys. Sin.* **53** 2046 (in Chinese)[楼智美 2004 物理学报 **53** 2046]

[9] Qiao Y F, Li R J, Ma Y S 2005 *Chin. Phys.* **14** 12

[10] Chen X W, Li Y M 2005 *Chin. Phys.* **14** 663

[11] Zhang Y 2005 *Acta Phys. Sin.* **54** 2980 (in Chinese)[张　毅 2005 物理学报 **54** 2980]

[12] Mei F X, Xu X J, Zhang Y F 2004 *Acta Mech. Sin.* **20** 668

[13] Xu X J, Qin M C, Mei F X 2005 *Chin. Phys.* **14** 1287
[14] Wu H B 2005 *Chin. Phys.* **14** 452
[15] Mei F X 2004 *Chin. Sci. Bull.* **47** 1544 (in Chinese) [梅凤翔 2004 科学通报 **47** 1544]
[16] Fang J H, Liao Y P, Peng Y 2004 *Chin. Phys.* **13** 1620

Lie-Mei symmetry of mechanical system in phase space

Fang Jian-Hui   Wang Peng   Ding Ning
(*College of Physics Science and Technology, China University of Petroleum (East China), Dongying* 257061, *China*)

(Received 18 October 2005; revised manuscript received 18 November 2005)

Abstract

In this paper, a new kind of symmetry and its conserved quantities of a mechanical system in the phase space are studied. The definition of this new symmetry, i.e., the Lie-Mei symmetry is presented, and the criterion of this new symmetry is also given. The generalized Hojman and the Mei conserved quantities of the Lie-Mei symmetry of the system are obtained. An example is given to illustrative the application of the result.

Keywords: phase space, mechanical system, Lie-Mei symmetry, conserved quantity
PACC: 0320