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度量算符对 Gauss 编织态的本征作用及自旋几何*

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对圈量子引力中标架度量矩阵算符对 Gauss 编织态的作用为本征作用,提供了完整的证明.求得了全部标架度量矩阵算符的表示矩阵,及其期望值.利用自旋几何定理,在内腿颜色 $k=0$ 和 $k=2$ 两种情况下,算得了 Gauss 编织态顶角毗邻的 4 条腿($P=1$)的相位位形切方向间的全部夹角,以及切矢量的长度.

关键词:度量算符的表示矩阵,度量期望值,切方向间夹角,切矢量长度

PACC:0460

1. 引 言

圈量子引力的重要成果之一,是给出了空间体积与面积量子化的描述[1—3].当物理学的考查从微观普朗克尺度出发向宏观发展时,时空的量子离散性将向半经典的连续性过度.实现这一过度的最好方法,目前认为是利用自旋网态的编织[4].用于编织空间的自旋网态(编织态),是作为体积与面积算符的本征态进行 3 维'累积'而编织空间的[5].在哈密顿约束下编织态演化的一维动力学经历,便是时间.时间也是量子化的,并且量子化的时间自身是有生衍历史的.而时空本身则是量子力学意义下态的动力学组合关系系统.文献[2,3]用代数方法给出了自旋网的腿'刺出'的曲面量子化面积与任意价顶角贡献的离散体积两者的统一手段描述.清除了一些文献中使用的方法的不统一以及结果上的不一致.

用态的位形进行编织是由文献[6]给出.该文明确了平坦空间的通常度量可通过几何算符的作用被赋予.后来研究发现,这种几何算符只能是面积与体积算符[1,4].

通过重叠不同基(态)的顶点,进而逐点编织空间区域的方法,即 Gauss 编织,近年来也得到了开展[7].时空做 $3+1$ 分解后得到的 3 维空间 Σ 中的区域 R 中的 Gauss 编织态通常写成 $W=\Pi_{\upsilon\in R}W_\upsilon$,而 $W_\upsilon=\sum_{p=0}^{\infty}C_p\psi_p$ 为在顶角 υ 的 Gauss 编织态.υ 为用以张成 W_υ 的基 ψ_p 的顶角.对于 Gauss 编织而言,在基 ψ_p 的顶角 υ 上可定义另一类几何算符,即标架度量.它可以给出被编织空间区域 R 的 Gauss 编织态的度量,从而使 Gauss 编织态自身成为具有度量的态.这对研究 Gauss 编织的进一步结果将具有意义.

Gauss 编织已有一些探讨[8,9],然而所见的某些结果并不够完整和正确[10].本文通过首尾一贯的计算,全面给出了标架度量矩阵算符对 Gauss 编织态(以基 ψ_p 代表)的作用为本征作用的完整证明,详细求得了度量矩阵算符的每一矩阵表示式,以及它们的所有期望值.同时,利用这些结果,按 Penrose 的自旋几何定理[11],计算了($p=1$ 峰值时内腿颜色 $k=0$ 和 2 总共这两种情况下)Gauss 编织态顶角处全部切方向间的夹角,以及切矢量的长度.结果表明,这一思路是可行的.

2. 度量矩阵算符对角分量 $\hat{M}(s_a,s_\alpha)$ 对顶角 ϕ_k 的本征作用

据作者所识,并无先验的理由保证,标架度量算符对自旋网态的作用均是本征作用.这里将通过计算,完成它对 Gauss 编织态本征作用的严格证明,并给出全部相关具体结果.

将基 ψ_p 及其展开后的顶角 υ,写成如下图形形式:

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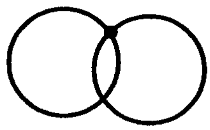
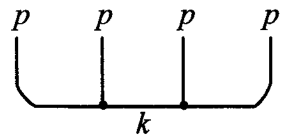


和 $N_k(p)$ 

$= \phi_k(p,p,p,p)$,

这里，$N_k(p) = N_k(p,p,p,p)$是其相伴顶角的正规化子. Gauss 编织的标架度量的有关证明将由正规化顶角 ϕ_k 开始. 作用在 ϕ_k 上的标架度量矩阵算符为

$$\hat{M}(s_\alpha, s_\beta)(\upsilon) = \frac{1}{2}[\hat{\Theta}^i_\alpha(\upsilon)\hat{\Theta}^i_\beta(\upsilon) + \alpha \leftrightarrow \beta], \quad (1)$$

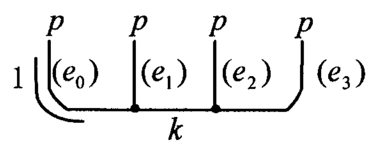
这里

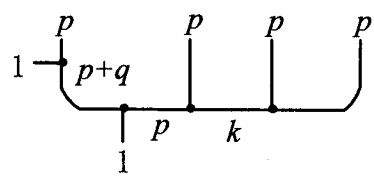
$$\hat{\Theta}^i_\alpha(\upsilon) = \frac{4}{\mathrm{i}\hbar k}\mathrm{Tr}(\tau^i h_\alpha \hat{V} h_\alpha^{-1}), \quad (2)$$

并且 holonomy$h_\alpha = h_{s_\alpha}$，s_α 是一端点在顶角 υ 上的线(段)，τ^i 为 $su(2)$生成元，$\hat{V}$ 为圈量子引力中的体积算符.

2.1. 算符 $\hat{M}(s_0, s_0)$对 ϕ_k 的本征作用

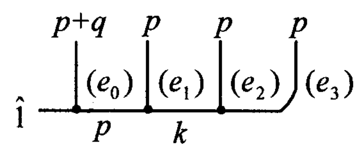
首先，计算 holonomyh_0^{-1}对 4 顶角 $\phi_k(p,p,p,p)$的作用，即

$h_0^{-1}\phi_k(p,p,p,p) = N_k(p)$ 

$= N_k(P)\sum_{q=\pm1}\gamma_q(P)$ 

, (3)

这里，$\gamma_+(p) = 1$，$\gamma_-(p) = -\frac{P}{P+1}$. 将体积算符 $\hat{V}$ 对(3)式最后一图的作用写成

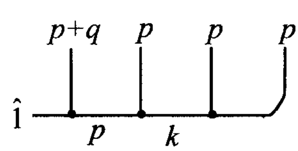
$\hat{V}h_0^{-1}\phi_k = N_k(p)\sum_{q=\pm1}\gamma_q(P)\hat{V}$ 

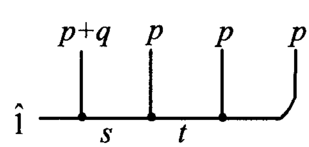
, (4)

并且，将颜色为 1 的小腿看成自由腿，且以$\hat{1}$记之，如(4)式所示，则体积算符 $\hat{V}$ 将不对该腿作用. $\hat{V}$ 对(4)式中的 5 顶角 ϕ_{pk}的作用将由下式给出：

$$\hat{V}\phi_{pk} = \sum_{s,t} V^{st}_{pk}\phi_{st}, \quad (5a)$$

这里

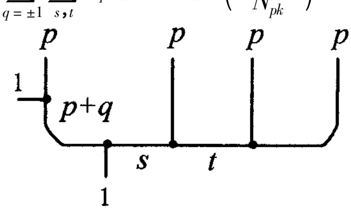
$N_{pk}(\hat{1}, p+q, p, p, p)\hat{V}$ 

$= \sum_{s,t} N_{st}(\hat{1}, p+q, p, p, p)V^{st}_{pk}$ 

(5b)

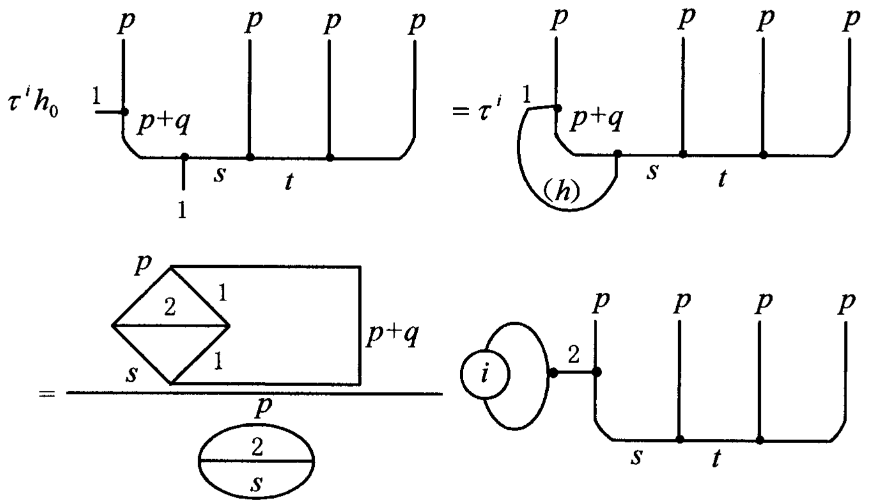
从而(4)式变为

$$\hat{V}h_0^{-1}\phi_k = \sum_{q=\pm1}\sum_{s,t}\gamma_q(p)N_k(p)\left(\frac{N_{st}V^{st}_{pk}}{N_{pk}}\right)(\hat{1}, p+q, p, p, p)$$



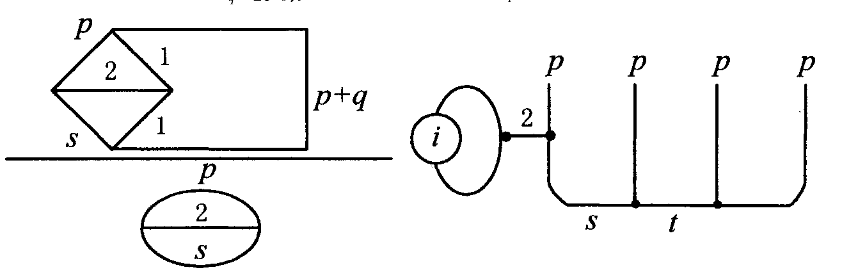
. (6)

现在考虑 $\tau^i h_0$ 对(6)式中的图的作用，有



将如上结果代入(2)式，得到

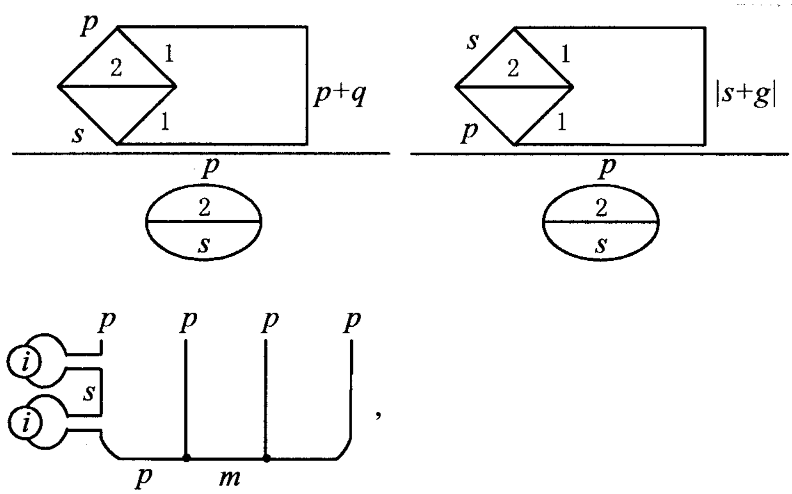
$$\hat{\Theta}_0^i \phi_k = \frac{4}{\mathrm{i}\hbar k}\sum_{q=\pm1}\sum_{s,t}\gamma_q(p)N_k(p)\left(\frac{N_{st}V_{pk}^{st}}{N_{pk}}\right)(\hat{1},p+q,p,p,p)$$



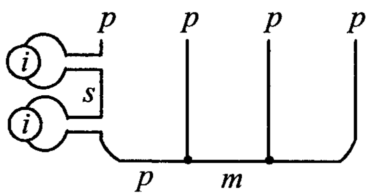
(7)

将 $\hat{\Theta}_0^i$ 对(7)式再施行一次作用，并将所得结果代入(1)式，可得

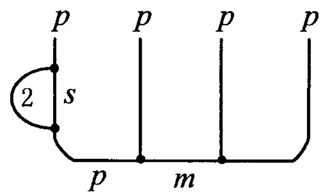
$$\hat{M}(s_0,s_0)\phi_k = -\frac{16}{(\hbar k)^2}\sum_{q=\pm1}\sum_{s,t}\sum_{g=\pm1}\sum_{m}\gamma_q(p)\gamma_g(s)\left(\frac{N_k}{N_m}\right)(p)$$
$$\times\left(\frac{N_{st}V_{pk}^{st}}{N_{pk}}\right)(\hat{1},p+q,p,p,p)\left(\frac{N_{pm}V_{st}^{pm}}{N_{st}}\right)(\hat{1},|s+g|,p,p,p)$$



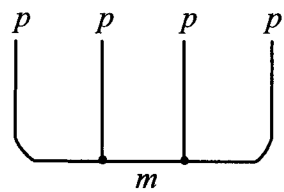
这里



$$= \frac{1}{2}$$



$$= \frac{1}{2}\frac{\theta(s,2,p)}{\Delta_p}$$



.

最后，将得到度量矩阵算符分量 $\hat{M}(s_0,s_0)$ 对态 ϕ_k 的作用为本征作用的如下表式：

$$\hat{M}(s_0,s_0)\phi_k = \sum_m M(s_0,s_0)_{(p)km}\phi_m, \tag{8a}$$

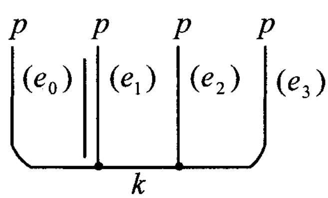
式中度量分量算符 $\hat{M}(s_0,s_0)$ 的表示矩阵元为

$$\begin{aligned} M(s_0,s_0)_{(p)km} = & -\frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{s,t}\sum_{g=\pm1}\gamma_q(p)\gamma_g(s)\left(\frac{N_k}{N_m}\right)(p) \\ & \times\left(\frac{N_{st}V_{pk}^{st}}{N_{pk}}\right)(\hat{1},p+q,p,p,p)\left(\frac{N_{pm}V_{st}^{pm}}{N_{st}}\right)(\hat{1},|s+g|,p,p,p) \\ & \times\frac{\mathrm{Tet}\begin{bmatrix} s & p & p+q \\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & s & |s+g| \\ 1 & 1 & 2\end{bmatrix}}{\Delta_p\theta(p,2,s)}. \end{aligned} \tag{8b}$$

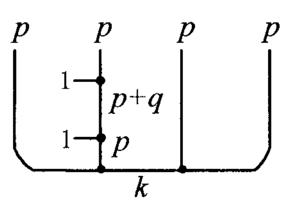
2.2. 度量矩阵算符其余对角分量对 ϕ_k 的本征作用

与前面类似，holonomy h_1^{-1} 对顶角 $\phi_k(p,p,p,p)$ 的作用可给出如下：

$$h_1^{-1}\phi_k(p,p,p,p) = N_k(p)$$

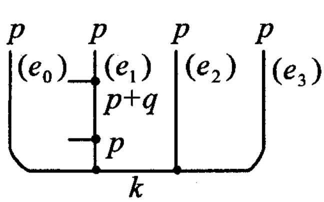


$$= N_k(p)\sum_{q=\pm1}\gamma_q(p)$$

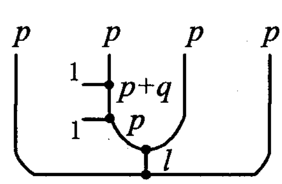


.

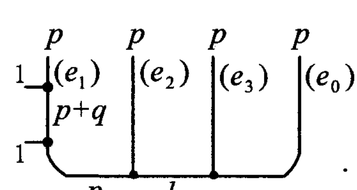
这里指出，为了实施体积算符 $\hat{V}$ 对上式的作用，必须在被作用的图形遵守一贯规则的形式下进行，故与其他文献（如文献[10]）不同，本文应用重耦定理[12]，将上式最后无法直接经受 $\hat{V}$ 的作用的图，变成如下等价形式：



$$= \sum_l \begin{Bmatrix} p & p & l \\ p & p & k \end{Bmatrix}$$



$$= \sum_l \begin{Bmatrix} p & p & l \\ p & p & k \end{Bmatrix}$$



. (9)

式中$\begin{Bmatrix} p & p & l \\ p & p & k \end{Bmatrix}$为 6-$j$ 记号.(9)式中最后一图,与(3)式最后一图的形式相当,这样才可以经受体积算符 $\hat{V}$ 的作用.从而,算符 $\hat{\Theta}_1^i$ 中其他因子对(9)式的作用结果与 $\hat{\Theta}_0^i$ 的作用是不同的,在最后结果中将出现 6-j 记号,它是在有关 5 顶角$(\hat{1},p+q,p,p,p)$的计算中出现的.从而算符 $\hat{M}(s_1,s_1)$对 ϕ_k 的本征作用将求得如下：

$$\hat{M}(s_1,s_1)\phi_k = \sum_m M(s_1,s_1)_{(p)km}\phi_m ,$$

式中表示矩阵元

$$\begin{aligned} M(s_1,s_1)_{(p)km} = &-\frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{s,t}\sum_{g=\pm1}\gamma_q(p)\gamma_g(s)\left(\frac{N_k}{N_m}\right)(p) \\ &\times\left(\sum_l \begin{Bmatrix} p & p & l \\ p & p & k \end{Bmatrix}\left(\frac{N_{st}V_{pl}^{st}}{N_{pl}}\right)(\hat{1},p+q,p,p,p)\right) \\ &\times\left(\frac{N_{pm}V_{st}^{pm}}{N_{st}}\right)(\hat{1},|s+g|,p,p,p)\frac{\mathrm{Tet}\begin{bmatrix} s & p & p+q \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & s & |s+g| \\ 1 & 1 & 2 \end{bmatrix}}{\Delta_p\theta(s,2,p)}. \end{aligned} \tag{10}$$

度量矩阵分量算符 $\hat{M}(s_2,s_2)$和 $\hat{M}(s_3,s_3)$对顶角 ϕ_k 的作用,可类似地求得,现给出如下：

$$\hat{M}(s_2,s_2)\phi_k = \sum_m M(s_2,s_2)_{(p)km}\phi_m ,$$

式中

$$\begin{aligned} M(s_2,s_2)_{(p)km} = &-\frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{s,t}\sum_{g=\pm1}\gamma_q(p)\gamma_g(s)\left(\frac{N_k}{N_m}\right)(p) \\ &\times\left(\sum_{l'} \begin{Bmatrix} p & p & l' \\ p & p & k \end{Bmatrix}\sum_l \begin{Bmatrix} p & p & l \\ p & p & l' \end{Bmatrix}\left(\frac{N_{st}V_{pl}^{st}}{N_{pl}}\right)(\hat{1},p+q,p,p,p)\right) \\ &\times\left(\frac{N_{pm}V_{st}^{pm}}{N_{st}}\right)(\hat{1},|s+g|,p,p,p)\frac{\mathrm{Tet}\begin{bmatrix} s & p & p+q \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & s & |s+g| \\ 1 & 1 & 2 \end{bmatrix}}{\Delta_p\theta(s,2,p)}; \end{aligned} \tag{11}$$

以及

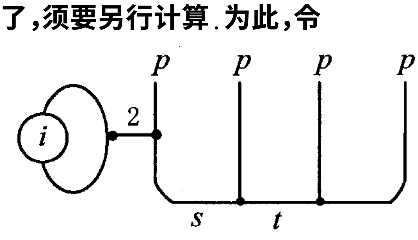
$$\hat{M}(s_3,s_3)\phi_k = \sum_m M(s_3,s_3)_{(p)km}\phi_m ,$$

式中

$$\begin{aligned} M(s_3,s_3)_{(p)km} = &-\frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{s,t}\sum_{g=\pm1}\gamma_q(p)\gamma_g(s)\left(\frac{N_k}{N_m}\right)(p) \\ &\times\left(\sum_{l'} \begin{Bmatrix} p & p & l' \\ p & p & k \end{Bmatrix}\sum_{l''} \begin{Bmatrix} p & p & l'' \\ p & p & l' \end{Bmatrix}\sum_l \begin{Bmatrix} p & p & l \\ p & p & l'' \end{Bmatrix}\left(\frac{N_{st}V_{pl}^{st}}{N_{pl}}\right)(\hat{1},p+q,p,p,p)\right) \\ &\times\left(\frac{N_{pm}V_{st}^{pm}}{N_{st}}\right)(\hat{1},|s+g|,p,p,p)\frac{\mathrm{Tet}\begin{bmatrix} s & p & p+q \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & s & |s+g| \\ 1 & 1 & 2 \end{bmatrix}}{\Delta_p\theta(s,2,p)}. \end{aligned} \tag{12}$$

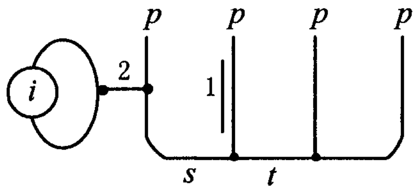
3. 度量矩阵算符非对角分量 $\hat{M}(s_\alpha,s_\beta)$对顶角 ϕ_k 的本征作用

算符 $\hat{M}(s_\alpha,s_\beta)(\alpha\neq\beta)$对顶角 ϕ_k 的作用,是通过在不同腿上的二次作用完成的,在计算上与节 1 有所不同.例如,$\hat{M}(s_0,s_1)$对 4 顶角 ϕ_k 的作用,其中 $\hat{\Theta}_0^i$ 对 ϕ_k 的作用将得到与(7)式相同的结果,然而,算符 $\hat{\Theta}_1^i$ 继续对(7)式的作用,就不再与前面相同了,须要另行计算.为此,令

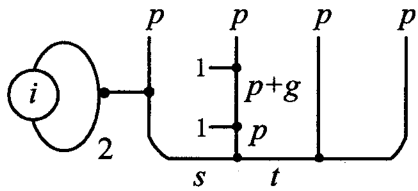
$F_{st}^i =$  ,

这样将得到 h_1^{-1} 对上图的作用为

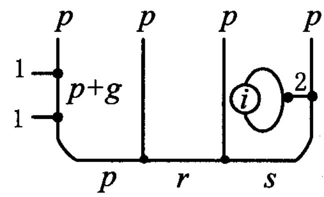
$$h_1^{-1}F_{st}^{i}=$$



$$=\sum_{g=\pm1}\gamma_g(p)$$



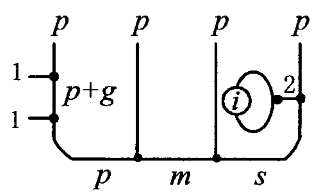
$$=\sum_{g=\pm1}\gamma_g(p)\left(\sum_r\begin{Bmatrix}s & p & r\\ p & p & t\end{Bmatrix}\right.$$



$$\left.\vphantom{\sum}\right).$$

上式在体积算符 $\hat{V}$ 作用下,变为

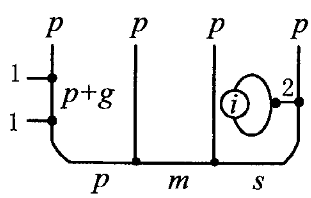
$$\hat{V}h_1^{-1}F_{st}^{i}=\sum_m\sum_{g=\pm1}\gamma_g(p)\left(\sum_r\begin{Bmatrix}p & p & r\\ p & p & t\end{Bmatrix}\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)(\hat{1},p+g,p,p,p)\right)$$



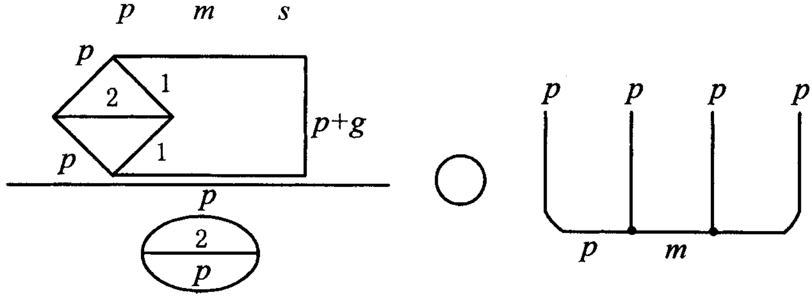
. (13)

(13)式的右端最后一个图在 $\tau^i h_1$ 作用下,将变为

$$\tau^i h_1$$

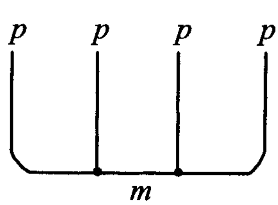


$$=\frac{1}{2}$$



(14)

$$=-\frac{\mathrm{Tet}\begin{bmatrix}p & p & p+g\\ 1 & 1 & 2\end{bmatrix}}{\theta(p,2,p)}$$



由(13)，(14)式，经整理后，可得

$$\hat{\Theta}_1^i\hat{\Theta}_0^i\phi_k = \frac{16}{(\hbar k)^2}\sum_{q=\pm1}\sum_{g=\pm1}\sum_t\sum_m \gamma_q(p)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p)\left(\frac{N_{pt}V_{pk}^{pt}}{N_{pk}}\right)(\hat{1},p+q,p,p,p)$$
$$\times\left(\sum_r\begin{Bmatrix}p & p & r\\ p & p & t\end{Bmatrix}\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)(\hat{1},p+g,p,p,p)\right)$$
$$\times\frac{\mathrm{Tet}\begin{bmatrix}p & p & p+q\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}p & p & p+g\\ 1 & 1 & 2\end{bmatrix}}{\theta(p,2,p)\theta(p,2,p)}\phi_m. \tag{15}$$

这样，$\hat{\Theta}_0^i\hat{\Theta}_1^i$ 对 ϕ_k 的作用结果，可类似地得到

$$\hat{\Theta}_0^i\hat{\Theta}_1^i\phi_k = \frac{16}{(\hbar k)^2}\sum_{q=\pm1}\sum_{g=\pm1}\sum_t\sum_m \gamma_q(p)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p)\left(\sum_l\begin{Bmatrix}p & p & l\\ p & p & k\end{Bmatrix}\right.$$
$$\times\left.\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1},p+q,p,p,p)\right)\left(\sum_{l''}\begin{Bmatrix}p & p & l''\\ p & p & t\end{Bmatrix}\right.$$
$$\times\left(\sum_{l'}\begin{Bmatrix}p & p & l'\\ p & p & l''\end{Bmatrix}\left(\sum_l\begin{Bmatrix}p & p & l\\ p & p & l'\end{Bmatrix}\left(\frac{N_{pm}V_{pl}^{pm}}{N_{pl}}\right)\right.\right.$$
$$\times\left.(\hat{1},p+g,p,p,p)\right)\frac{\mathrm{Tet}\begin{bmatrix}p & p & p+q\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}p & p & p+g\\ 1 & 1 & 2\end{bmatrix}}{\theta(p,2,p)\theta(p,2,p)}\phi_m.$$

将上式及(15)式代入(1)式，将得到如下本征作用：

$$\hat{M}(s_0,s_1)\phi_k = \sum_m M(s_0,s_1)_{(p)km}\phi_m,$$

式中表示矩阵元

$$M(s_0,s_1)_{(p)km} = \frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{g=\pm1}\sum_t \gamma_q(p)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p)$$
$$\times\left\{\left(\frac{N_{pt}V_{pk}^{pt}}{N_{pk}}\right)(\hat{1},p+q,p,p,p)\left(\sum_r\begin{Bmatrix}p & p & r\\ p & p & t\end{Bmatrix}\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)(\hat{1},p+g,p,p,p)\right)\right.$$
$$+\left(\sum_l\begin{Bmatrix}p & p & l\\ p & p & k\end{Bmatrix}\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1},p+q,p,p,p)\right)$$
$$\times\left(\sum_{l''}\begin{Bmatrix}p & p & l''\\ p & p & t\end{Bmatrix}\sum_{l'}\begin{Bmatrix}p & p & l'\\ p & p & l''\end{Bmatrix}\sum_{l'}\begin{Bmatrix}p & p & l\\ p & p & l'\end{Bmatrix}\left(\frac{N_{pm}V_{pl}^{pm}}{N_{pl}}\right)\right.$$
$$\left.\left.\times(\hat{1},p+g,p,p,p)\right)\right\}\frac{\mathrm{Tet}\begin{bmatrix}p & p & p+q\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}p & p & p+g\\ 1 & 1 & 2\end{bmatrix}}{\theta(p,2,p)\theta(p,2,p)}. \tag{16}$$

可类似地得到算符 $\hat{M}(s_0,s_2)$和 $\hat{M}(s_0,s_3)$的本征作用，现直接给出如下：

$$\hat{M}(s_0,s_2)\phi_k = \sum_m M(s_0,s_2)_{(p)km}\phi_m,$$

这里

$$M(s_0,s_2)_{(p)km} = \frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{g=\pm1}\sum_t \gamma_q(p)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p)$$
$$\times\left\{\left(\frac{N_{pt}V_{pk}^{pt}}{N_{pk}}\right)(\hat{1},p+q,p,p,p)\left(\sum_{r'}\begin{Bmatrix}p & p & r'\\ p & p & t\end{Bmatrix}\sum_r\begin{Bmatrix}p & p & r\\ p & p & r'\end{Bmatrix}\right.\right.$$
$$\times\left.\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)(\hat{1},p+g,p,p,p)\right)+\left(\sum_{l'}\begin{Bmatrix}p & p & l'\\ p & p & k\end{Bmatrix}\sum_l\begin{Bmatrix}p & p & l\\ p & p & l'\end{Bmatrix}\right.$$

$$
\times\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1},p+q,p,p,p)\Bigg)\Bigg(\sum_{l'}\begin{Bmatrix}p & p & l'\\ p & p & t\end{Bmatrix}\sum_{l}\begin{Bmatrix}p & p & l\\ p & p & l'\end{Bmatrix}
$$

$$
\times\left(\frac{N_{pm}V_{pl}^{pm}}{N_{pl}}\right)(\hat{1},p+g,p,p,p)\Bigg)\Bigg\}\frac{\mathrm{Tet}\begin{bmatrix}p & p & p+q\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}p & p & p+g\\ 1 & 1 & 2\end{bmatrix}}{\theta(p,2,p)\theta(p,2,p)}; \tag{17}
$$

以及

$$
\hat{M}(s_0,s_3)\phi_k=\sum_m M(s_0,s_3)_{(p)km}\phi_m,
$$

这里

$$
M(s_0,s_3)_{(p)km}=\frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{g=\pm1}\sum_t\gamma_q(p)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p)\Bigg\{\left(\frac{N_{pt}V_{pk}^{pt}}{N_{pk}}\right)(\hat{1},p+q,p,p,p)
$$

$$
\times\Bigg(\sum_{r'}\begin{Bmatrix}p & p & r'\\ p & p & t\end{Bmatrix}\sum_{r''}\begin{Bmatrix}p & p & r''\\ p & p & r'\end{Bmatrix}\sum_{r}\begin{Bmatrix}p & p & r\\ p & p & r''\end{Bmatrix}\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)(\hat{1},p+g,p,p,p)\Bigg)
$$

$$
+\Bigg(\sum_{l'}\begin{Bmatrix}p & p & l'\\ p & p & k\end{Bmatrix}\sum_{l''}\begin{Bmatrix}p & p & l''\\ p & p & l'\end{Bmatrix}\sum_{l}\begin{Bmatrix}p & p & l\\ p & p & l''\end{Bmatrix}\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1},p+q,p,p,p)\Bigg)
$$

$$
\times\Bigg(\sum_{l}\begin{Bmatrix}p & p & l\\ p & p & t\end{Bmatrix}\left(\frac{N_{pm}V_{pl}^{pm}}{N_{pl}}\right)(\hat{1},p+g,p,p,p)\Bigg)\Bigg\}
$$

$$
\times\frac{\mathrm{Tet}\begin{bmatrix}p & p & p+q\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}p & p & p+g\\ 1 & 1 & 2\end{bmatrix}}{\theta(p,2,p)\theta(p,2,p)}. \tag{18}
$$

算符 $\hat{M}(s_1,s_2)$,$\hat{M}(s_1,s_3)$和 $\hat{M}(s_2,s_3)$对顶角 ϕ_k 的作用可进而类似地求得,现将其结果给出如下:

$$
\hat{M}(s_1,s_2)\phi_k=\sum_m M(s_1,s_2)_{(p)km}\phi_m,
$$

式中

$$
M(s_1,s_2)_{(p)km}=\frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{g=\pm1}\sum_t\gamma_q(p)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p)
$$

$$
\times\Bigg\{\Bigg(\sum_{l}\begin{Bmatrix}p & p & l\\ p & p & k\end{Bmatrix}\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1},p+q,p,p,p)\Bigg)\Bigg(\sum_{r}\begin{Bmatrix}p & p & r\\ p & p & t\end{Bmatrix}
$$

$$
\times\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)(\hat{1},p+g,p,p,p)\Bigg)+\Bigg(\sum_{l'}\begin{Bmatrix}p & p & l'\\ p & p & k\end{Bmatrix}\sum_{l}\begin{Bmatrix}p & p & l\\ p & p & l'\end{Bmatrix}
$$

$$
\times\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1},p+q,p,p,p)\Bigg)\Bigg(\sum_{l''}\begin{Bmatrix}p & p & l''\\ p & p & t\end{Bmatrix}\sum_{l'}\begin{Bmatrix}p & p & l'\\ p & p & l''\end{Bmatrix}\sum_{l}\begin{Bmatrix}p & p & l\\ p & p & l'\end{Bmatrix}
$$

$$
\times\left(\frac{N_{pm}V_{pl}^{pm}}{N_{pl}}\right)(\hat{1},p+g,p,p,p)\Bigg)\Bigg\}\frac{\mathrm{Tet}\begin{bmatrix}p & p & p+q\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}p & p & p+g\\ 1 & 1 & 2\end{bmatrix}}{\theta(p,2,p)\theta(p,2,p)}; \tag{19}
$$

$$
\hat{M}(s_1,s_3)\phi_k=\sum_m M(s_1,s_3)_{(p)km}\phi_m,
$$

式中

$$
M(s_1,s_2)_{(p)km}=\frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{g=\pm1}\sum_t\gamma_q(q)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p)
$$

$$
\times\Bigg\{\Bigg(\sum_{l}\begin{Bmatrix}p & p & l\\ p & p & k\end{Bmatrix}\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1},p+q,p,p,p)\Bigg)
$$

$$\times\left(\sum_{r'}\begin{Bmatrix} p & p & r' \\ p & p & t \end{Bmatrix}\sum_{r}\begin{Bmatrix} p & p & r \\ p & p & r' \end{Bmatrix}\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)\hat{(1},p+g,p,p,p)\right)$$

$$+\left(\sum_{l'}\begin{Bmatrix} p & p & l' \\ p & p & k \end{Bmatrix}\sum_{l''}\begin{Bmatrix} p & p & l'' \\ p & p & l' \end{Bmatrix}\sum_{l}\begin{Bmatrix} p & p & l \\ p & p & l'' \end{Bmatrix}\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)\hat{(1},p+q,p,p,p)\right)$$

$$\times\left(\sum_{r'}\begin{Bmatrix} p & p & r' \\ p & p & t \end{Bmatrix}\sum_{r}\begin{Bmatrix} p & p & r \\ p & p & r' \end{Bmatrix}\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)\hat{(1},p+g,p,p,p)\right)\Bigg\}$$

$$\times\frac{\mathrm{Tet}\begin{bmatrix} p & p & p+q \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & p & p+g \\ 1 & 1 & 2 \end{bmatrix}}{\theta(p,2,p)\,\theta(p,2,p)}; \tag{20}$$

以及

$$\hat{M}(s_2,s_3)\phi_k = \sum_m M(s_2,s_3)_{(p)\,km}\phi_m,$$

式中

$$M(s_2,s_3)_{(p)\,km} = \frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{g=\pm1}\sum_{t}\gamma_q(p)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p)$$

$$\times\Bigg\{\left(\sum_{l'}\begin{Bmatrix} p & p & l' \\ p & p & k \end{Bmatrix}\sum_{l'}\begin{Bmatrix} p & p & l \\ p & p & l' \end{Bmatrix}\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)\hat{(1},p+q,p,p,p)\right)$$

$$\times\left(\sum_{r}\begin{Bmatrix} p & p & r \\ p & p & t \end{Bmatrix}\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)\hat{(1},p+g,p,p,p)\right)$$

$$+\left(\sum_{l'}\begin{Bmatrix} p & p & l' \\ p & p & k \end{Bmatrix}\sum_{l''}\begin{Bmatrix} p & p & l'' \\ p & p & l' \end{Bmatrix}\sum_{l}\begin{Bmatrix} p & p & l \\ p & p & l'' \end{Bmatrix}\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)\hat{(1},p+q,p,p,p)\right)$$

$$\times\left(\sum_{r''}\begin{Bmatrix} p & p & r'' \\ p & p & t \end{Bmatrix}\sum_{r'}\begin{Bmatrix} p & p & r' \\ p & p & r'' \end{Bmatrix}\sum_{r}\begin{Bmatrix} p & p & r \\ p & p & r' \end{Bmatrix}\right.$$

$$\left.\times\left(\frac{N_{pm}V_{pr}^{n\,pm}}{N_{pr}}\right)\hat{(1},p+g,p,p,p)\right)\Bigg\}\frac{\mathrm{Tet}\begin{bmatrix} p & p & p+q \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & p & p+g \\ 1 & 1 & 2 \end{bmatrix}}{\theta(p,2,p)\,\theta(p,2,p)}. \tag{21}$$

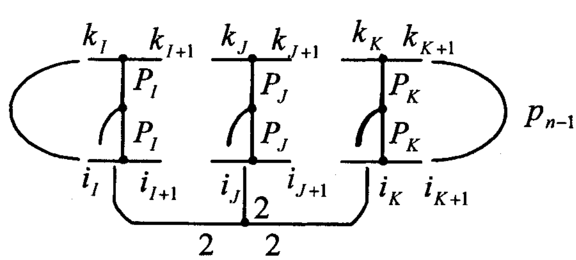
4. 体积算符的通用重耦矩阵表式

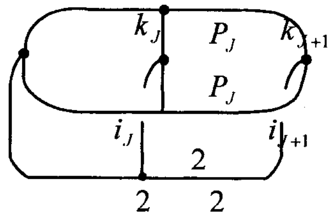
体积算符 $\hat{V}$ 对 5 顶角 ϕ_{pk} 的作用将用(5)式实现，(5)式体积矩阵元 V_{pk}^{st} 本文由下式给出[12]：

$$V_{pk}^{st} = \frac{(\hbar k)^{\frac{3}{2}}}{4}\sqrt{\sum_{[IJK]}W_{[IJK]}{}_{pk}^{st}}, \tag{22}$$

这里 $W_{[IJK]}{}_{pk}^{st}$ 为重耦矩阵元. 为了求得该重耦矩阵元，我们将首先引入用于任意价顶角的体积算符的通用重耦矩阵公式如下：

$$W_{[IJK]}{}_{i}^{k} = N_iN_kP_IP_JP_K\cdot p_0$$



$$= N_{\boldsymbol{i}} N_{\boldsymbol{k}} P_I P_J P_K \alpha_i \alpha_{ii} \alpha_{iii} \alpha_{iv}$$ 

$$= N_{\boldsymbol{i}} N_{\boldsymbol{k}} P_I P_J P_K \alpha_i \alpha_{ii} \alpha_{iii} \alpha_{iv} \begin{Bmatrix} k_J & P_J & k_{J+1} \\ i_J & P_J & i_{J+1} \\ 2 & 2 & 2 \end{Bmatrix}, \tag{23}$$

式中 $N_{\boldsymbol{i}}$，$N_{\boldsymbol{k}}$ 为正规化子，$\boldsymbol{i} = i_2, \cdots, i_{n-2}$，$\boldsymbol{k} = k_2, \cdots, k_{n-2}$. α_i，α_{ii}，α_{iii}，α_{iv} 给出如下：

$$\alpha_i = -\lambda^{i_{I+1}^2}_{k_{I+1}} \left(\prod_{x=2}^{I} \delta^{k_x}_{i_x} \right) \frac{\theta(p_0, p_1, i_2)}{\Delta_{i_2}} \left(\prod_{x=2}^{I-1} \frac{\theta(i_x, p_x, i_{x+1})}{\Delta_{i_{2+1}}} \right) \frac{\mathrm{Tet}\begin{bmatrix} k_{I+1} & i_{I+1} & i_I \\ p_I & p_I & 2 \end{bmatrix}}{\theta(k_{I+1}, i_{I+1}, 2)}, \tag{24}$$

$$\alpha_{ii} = \prod_{x=I+1}^{J-1} \frac{\mathrm{Tet}\begin{bmatrix} k_x & k_{x+1} & 2 \\ i_{x+1} & i_x & p_x \end{bmatrix}}{\theta(k_{x+1}, i_{x+1}, 2)}, \tag{25}$$

$$\alpha_{iii} = \frac{\lambda^{i_K^2}_{k_K}}{\lambda^{i_{J+1}^2}_{k_{J+1}}} \left[\prod_{x=J+1}^{k-1} \frac{\mathrm{Tet}\begin{bmatrix} k_x & k_{x+1} & 2 \\ i_{x+1} & i_x & p_x \end{bmatrix}}{\theta(k_x, i_x, 2)} \right], \tag{26}$$

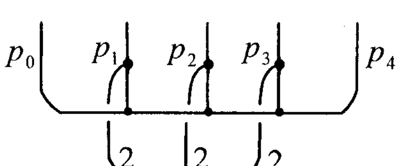
$$\alpha_{iv} = \left(\prod_{x=K+1}^{n-2} \delta^{k_x}_{i_x} \right) \left(\prod_{x=K+1}^{n-3} \frac{\theta(i_x, p_x, i_{x+1})}{\Delta_{i_x}} \right) \frac{\theta(i_{n-2}, p_{n-2}, p_{n-1})}{\Delta_{i_{n-2}}} \frac{\mathrm{Tet}\begin{bmatrix} i_K & k_K & i_{k+1} \\ p_K & p_K & 2 \end{bmatrix}}{\theta(k_K, i_K, 2)}. \tag{27}$$

5. 用于一般 5 顶角计算的重耦矩阵表式

在本文情况下，度量算符中所含有的体积算符将作用在 5 顶角上，而且 5 顶角中颜色为 1 的小腿 $\hat{1}$ 将为 $\hat{V}$ 视而不见，所以 $\hat{V}$ 只有 4 个抓的三元组施加在这种一般 5 顶角上，这些三元组将是“123”，“124”，“134”和“234”. 下面将计算体积算符 $\hat{V}$ 通过如上4 个三元组对一般 5 顶角作用而得的重耦矩阵.

5.1. $W_{[123]}{}^{k}_{i}$

此种情况下，(23) 式中的 $I = 1$，$J = 2$，$K = 3$，此时的抓法将由下图给出：



, (28)

且有 $\alpha_{ii} = \alpha_{iii} = 1$，$\boldsymbol{i} = i_1 i_2$，$\boldsymbol{k} = k_1 k_2$. 故由 (23) 式，有

$$W_{[123]}{}^{\boldsymbol{k}}_{\boldsymbol{i}} = N_{\boldsymbol{i}} N_{\boldsymbol{k}} P_1 P_2 P_3 \alpha_i \alpha_{iv} \begin{Bmatrix} k_2 & p_2 & k_3 \\ i_2 & p_2 & i_3 \\ 2 & 2 & 2 \end{Bmatrix}, \tag{29}$$

式中

$$N_{\boldsymbol{k}}(p_0, \cdots, p_4) = \sqrt{\frac{\prod_{x=2}^{3} \Delta_{k_x}}{\prod_{x=1}^{3} \theta(k_x, p_x, k_{x+1})}}, \tag{30}$$

$$N_{\boldsymbol{i}}(p_0, \cdots, p_4) = \sqrt{\frac{\prod_{x=2}^{3} \Delta_{i_x}}{\prod_{x=1}^{3} \theta(i_x, p_x, k_{x+1})}}, \tag{31}$$

$$\alpha_i = -\lambda^{i_2^2}_{k_2} \frac{\mathrm{Tet}\begin{bmatrix} k_2 & i_2 & i_1 \\ p_1 & p_1 & 2 \end{bmatrix}}{\theta(i_2, 2, k_2)} = -\frac{\mathrm{Tet}\begin{bmatrix} k_2 & i_2 & i_1 \\ p_1 & p_1 & 2 \end{bmatrix}}{\theta(i_2, 2, k_2)} \lambda^{i_2^2}_{k_2}, \tag{32}$$

$$\alpha_{iv} = \frac{\mathrm{Tet}\begin{bmatrix} i_3 & k_3 & i_4 \\ p_3 & p_3 & 2 \end{bmatrix}}{\theta(k_3, 2, i_3)}. \tag{33}$$

将 (30)—(33) 式代入 (29) 式，将得到抓三元组 [123]

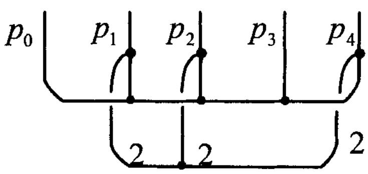
对应的重耦矩阵如下：

$$W_{[123]}{}_{i}^{k}=\sqrt{\frac{\Delta_{i_2}\Delta_{i_3}\Delta_{k_2}\Delta_{k_3}}{\theta(i_1,p_1,i_2)\theta(i_2,p_2,i_3)\theta(i_3,p_3,i_4)\theta(k_1,p_1,k_2)\theta(k_2,p_2,k_3)\theta(k_3,p_3,k_4)}}(-1)\lambda^{i_2 2}_{k_2} \times p_1p_2p_3\frac{\mathrm{Tet}\begin{bmatrix}k_2 & i_2 & i_1\\ p_1 & p_1 & 2\end{bmatrix}}{\theta(i_2,2,k_2)}\frac{\mathrm{Tet}\begin{bmatrix}i_3 & k_3 & i_4\\ p_3 & p_3 & 2\end{bmatrix}}{\theta(i_3,2,k_3)}\begin{Bmatrix}k_2 & p_2 & k_3\\ i_2 & p_2 & i_3\\ 2 & 2 & 2\end{Bmatrix}, \tag{34}$$

式中 $i_1=k_1=p_0, i_4=k_4=p_4$.

5.2. $W_{[124]}{}_{i}^{k}$

此时，因 $I=1$，$J=2$，$K=4$，相应的抓法图为



.

因此时有 $\alpha_{ii}=\alpha_{iv}=1$，对于抓的三元组[124]，其重耦矩阵为

$$W_{[124]}{}_{i}^{k}=N_iN_kP_1P_2P_4\alpha_i\alpha_{iii}\begin{Bmatrix}k_2 & p_2 & k_3\\ i_2 & p_2 & i_3\\ 2 & 2 & 2\end{Bmatrix}, \tag{35}$$

式中

$$\alpha_i=\frac{\mathrm{Tet}\begin{bmatrix}k_2 & i_2 & i_1\\ p_1 & p_1 & 2\end{bmatrix}}{\theta(i_2,2,k_2)}(-\lambda^{i_2 2}_{k_2}), \tag{36}$$

$$\alpha_{iii}=\frac{\lambda^{i_4 2}_{k_4}}{\lambda^{i_3 2}_{k_3}}\frac{\mathrm{Tet}\begin{bmatrix}k_3 & k_4 & 2\\ i_4 & i_3 & p_3\end{bmatrix}}{\theta(k_3,i_3,2)}=\frac{\mathrm{Tet}\begin{bmatrix}k_3 & k_4 & 2\\ i_4 & i_3 & p_3\end{bmatrix}}{\theta(k_3,i_3,2)}\frac{-1}{\lambda^{i_3 2}_{k_3}}. \tag{37}$$

将(30)，(31)，(36)和(37)式代入(35)式，将有

$$W_{[124]}{}_{i}^{k}=\sqrt{\frac{\Delta_{i_2}\Delta_{i_3}\Delta_{k_2}\Delta_{k_3}}{\theta(i_1,p_1,i_2)\theta(i_2,p_2,i_3)\theta(i_3,p_3,i_4)\theta(k_1,p_1,k_2)\theta(k_2,p_2,k_3)\theta(k_3,p_3,k_4)}}\frac{(-1)^2\lambda^{i_2 2}_{k_2}}{\lambda^{i_3 2}_{k_3}} \times p_1p_2p_4\frac{\mathrm{Tet}\begin{bmatrix}k_2 & i_2 & i_1\\ p_1 & p_1 & 2\end{bmatrix}}{\theta(i_2,2,k_2)}\frac{\mathrm{Tet}\begin{bmatrix}k_3 & k_4 & 2\\ i_4 & i_3 & p_3\end{bmatrix}}{\theta(k_3,i_3,2)}\begin{Bmatrix}k_2 & p_2 & k_3\\ i_2 & p_2 & i_3\\ 2 & 2 & 2\end{Bmatrix}, \tag{38}$$

5.3. $W_{[234]}{}_{i}^{k}$ **和** $W_{[134]}{}_{i}^{k}$

通过类似地计算，这两个重耦矩阵亦可以得到，现将其结果直接给出如下：

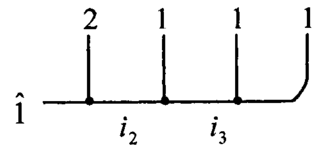
$$W_{[234]}{}_{i}^{k}=\sqrt{\frac{\Delta_{i_2}\Delta_{i_3}\Delta_{k_2}\Delta_{k_3}}{\theta(i_1,p_1,i_2)\theta(i_2,p_2,i_3)\theta(i_3,p_3,i_4)\theta(k_1,p_1,k_2)\theta(k_2,p_2,k_3)\theta(k_3,p_3,k_4)}} \times p_2p_3p_4\delta^{k_2}_{i_2}\frac{\theta(p_0,p_1,i_2)}{\Delta_{i_2}}\frac{\mathrm{Tet}\begin{bmatrix}k_3 & i_3 & i_2\\ p_2 & p_2 & 2\end{bmatrix}}{\theta(k_3,i_3,2)}\begin{Bmatrix}k_3 & p_3 & k_4\\ i_3 & p_3 & i_4\\ 2 & 2 & 2\end{Bmatrix}(-1)\lambda^{i_3 2}_{k_3}, \tag{39}$$

和

$$W_{[134]}{}_{i}^{k}=\sqrt{\frac{\Delta_{i_2}\Delta_{i_3}\Delta_{k_2}\Delta_{k_3}}{\theta(i_1,p_1,i_2)\theta(i_2,p_2,i_3)\theta(i_3,p_3,i_4)\theta(k_1,p_1,k_2)\theta(k_2,p_2,k_3)\theta(k_3,p_3,k_4)}} \times p_1p_3p_4(-1)\lambda^{i_2 2}_{k_2}\frac{\mathrm{Tet}\begin{bmatrix}k_2 & i_2 & i_1\\ p_1 & p_1 & 2\end{bmatrix}}{\theta(i_2,2,k_2)}\frac{\mathrm{Tet}\begin{bmatrix}k_2 & i_3 & 2\\ i_3 & i_2 & p_2\end{bmatrix}}{\theta(k_3,i_3,2)}\begin{Bmatrix}k_3 & p_3 & k_4\\ i_3 & p_3 & i_4\\ 2 & 2 & 2\end{Bmatrix}, \tag{40}$$

6.5 顶角$(\hat{1},2,1,1,1)$的重耦矩阵元

对 Gauss 编织而言，$p=1$ 和 $p=2$ 将给出用基 ψ_p 进行顶角编织时的峰值. 当 $p>2$ 时，对应的 ψ_p 将因其相应系数 C_p 的急剧减小，而对编织起的作用极大地减弱. 本文将给出 $p=1$ 时的有关证明，p 为其余值时的证明可类似地进行. 当 $p=1$ 时，由自旋网 3 顶角相容原理可知，此时有 $k=0$ 和 $k=2$ 两种情况，本文(与其他文献不同)将全部计算之. 在 $p=1$ 情况下，本文出现的 5 顶角$(\hat{1},p+q,1,1,1)$和$(\hat{1},|s+g|,1,1,1)$均将只有三种可能：$(\hat{1},0,1,1,1)$，$(\hat{1},2,1,1,1)$和$(\hat{1},4,1,1,1)$. 由于顶角$(\hat{1},4,1,1,1)$在算符 $\hat{V}$ 作用下为零，故下面将只需计算顶角$(\hat{1},0,1,1,1)$和$(\hat{1},2,1,1,1)$对应的重耦矩阵. 对于顶角$(\hat{1},2,1,1,1)$，它的图为



. (41)

6.1. 重耦矩阵 $W_{[123]}{}^{k_2k_3}_{i_2i_3}$ 的矩阵元

由于图(41)中的双重内腿 i_2i_3 的颜色组合可取为 10，12 和 32 三种，利用(34)，经过计算后，可得到本文所用的该重耦矩阵的如下矩阵元：

$$W_{[123]}{}^{10}_{10}=\sqrt{\frac{\Delta_1\Delta_0\Delta_1\Delta_0}{\theta(1,2,1)\theta(1,1,0)\theta(0,1,1)\theta(1,2,1)\theta(1,1,0)\theta(0,1,1)}}(-1)^2\times 2\frac{\mathrm{Tet}\begin{bmatrix}1&1&1\\2&2&2\end{bmatrix}}{\theta(1,2,1)}\frac{\mathrm{Tet}\begin{bmatrix}0&0&1\\1&1&2\end{bmatrix}}{\theta(0,2,0)}\begin{Bmatrix}1&1&0\\1&1&0\\2&2&2\end{Bmatrix}=0,$$

$$W_{[123]}{}^{12}_{10}=\frac{\sqrt{\Delta_1\Delta_2\Delta_1\Delta_0}}{|\theta(1,2,1)\theta(1,1,2)\theta(1,1,0)|}2\frac{\mathrm{Tet}\begin{bmatrix}1&1&1\\2&2&2\end{bmatrix}}{\theta(1,2,1)}\frac{\mathrm{Tet}\begin{bmatrix}0&2&1\\1&1&2\end{bmatrix}}{\theta(2,2,0)}(-1)^2\begin{Bmatrix}1&1&2\\1&1&0\\2&2&2\end{Bmatrix}=\frac{2}{3}\sqrt{3},$$

$$W_{[123]}{}^{32}_{10}=\frac{\sqrt{\Delta_1\Delta_0\Delta_3\Delta_2}}{|\theta(1,2,1)\theta(1,1,0)\theta(1,2,3)|}2\frac{\mathrm{Tet}\begin{bmatrix}3&1&1\\2&2&2\end{bmatrix}}{\theta(1,2,3)}\frac{\mathrm{Tet}\begin{bmatrix}0&2&1\\1&1&2\end{bmatrix}}{\theta(2,2,0)}(-1)^2\begin{Bmatrix}3&1&2\\1&1&0\\2&2&2\end{Bmatrix}=-\frac{\sqrt{6}}{6},$$

$$W_{[123]}{}^{10}_{12}=\frac{\sqrt{\Delta_1\Delta_2\Delta_1\Delta_0}}{|\theta(1,2,1)\theta(1,1,2)\theta(1,1,0)|}2\frac{\mathrm{Tet}\begin{bmatrix}1&1&1\\2&2&2\end{bmatrix}}{\theta(1,2,1)}\frac{\mathrm{Tet}\begin{bmatrix}2&0&1\\1&1&2\end{bmatrix}}{\theta(0,2,2)}(-1)^2\begin{Bmatrix}1&1&0\\1&1&2\\2&2&2\end{Bmatrix}=-\frac{2}{3}\sqrt{3},$$

$$W_{[123]}{}^{12}_{12}=\frac{\sqrt{\Delta_1\Delta_2\Delta_1\Delta_2}}{|\theta(1,2,1)\theta(1,1,2)\theta(2,1,1)|}2\frac{\mathrm{Tet}\begin{bmatrix}1&1&1\\2&2&2\end{bmatrix}}{\theta(1,2,1)}\frac{\mathrm{Tet}\begin{bmatrix}2&2&1\\1&1&2\end{bmatrix}}{\theta(2,2,2)}(-1)^2\begin{Bmatrix}1&1&2\\1&1&2\\2&2&2\end{Bmatrix}=0,$$

$$W_{[123]}{}^{32}_{12}=\frac{\sqrt{\Delta_3\Delta_2\Delta_1\Delta_2}}{|\theta(1,1,2)\theta(1,1,2)\theta(1,2,3)|}2\frac{\mathrm{Tet}\begin{bmatrix}3&1&1\\2&2&2\end{bmatrix}}{\theta(1,2,3)}\frac{\mathrm{Tet}\begin{bmatrix}2&2&1\\1&1&2\end{bmatrix}}{\theta(2,2,2)}\begin{Bmatrix}3&1&2\\1&1&2\\2&2&2\end{Bmatrix}(-1)^2$$

$$= -\frac{\sqrt{2}}{2},$$

$$W_{[123]}{}^{10}_{32} = \frac{\sqrt{\Delta_1\Delta_0\Delta_3\Delta_2}}{|\theta(1,1,2)\theta(1,1,0)\theta(1,2,3)|}2\frac{\mathrm{Tet}\begin{bmatrix}1&3&1\\2&2&2\end{bmatrix}}{\theta(3,2,1)}\frac{\mathrm{Tet}\begin{bmatrix}2&0&1\\1&1&2\end{bmatrix}}{\theta(0,2,2)}(-1)^2\begin{Bmatrix}1&1&0\\3&1&2\\2&2&2\end{Bmatrix}$$

$$= \frac{\sqrt{6}}{6},$$

$$W_{[123]}{}^{12}_{32} = \frac{\sqrt{\Delta_1\Delta_2\Delta_3\Delta_2}}{|\theta(1,1,2)\theta(1,1,2)\theta(1,2,3)|}2\frac{\mathrm{Tet}\begin{bmatrix}1&3&1\\2&2&2\end{bmatrix}}{\theta(3,2,1)}\frac{\mathrm{Tet}\begin{bmatrix}2&2&1\\1&1&2\end{bmatrix}}{\theta(2,2,2)}(-1)^2\begin{Bmatrix}1&1&2\\3&1&2\\2&2&2\end{Bmatrix}$$

$$= \frac{\sqrt{2}}{2},$$

$$W_{[123]}{}^{32}_{32} = \frac{\sqrt{\Delta_3\Delta_2\Delta_3\Delta_2}}{|\theta(1,2,3)\theta(1,2,3)\theta(1,1,2)|}2\frac{\mathrm{Tet}\begin{bmatrix}3&3&1\\2&2&2\end{bmatrix}}{\theta(3,2,3)}\frac{\mathrm{Tet}\begin{bmatrix}2&2&1\\1&1&2\end{bmatrix}}{\theta(2,2,2)}(-1)^2\begin{Bmatrix}3&1&2\\3&1&2\\2&2&2\end{Bmatrix}$$

$$= 0.$$

从而,相应的重耦矩阵可写成

$$W_{[123]}{}^{k_2k_3}_{i_2i_3} = \begin{bmatrix} W_{[123]}{}^{10}_{10} & W_{[123]}{}^{12}_{10} & W_{[123]}{}^{32}_{10} \\ W_{[123]}{}^{10}_{12} & W_{[123]}{}^{12}_{12} & W_{[123]}{}^{32}_{12} \\ W_{[123]}{}^{10}_{32} & W_{[123]}{}^{12}_{32} & W_{[123]}{}^{32}_{32} \end{bmatrix} = \begin{bmatrix} 0 & \frac{2\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} \\ -\frac{2}{3}\sqrt{3} & 0 & -\sqrt{2}/2 \\ \sqrt{6}/6 & \sqrt{2}/2 & 0 \end{bmatrix}. \tag{42}$$

6.2. 其余重耦矩阵的矩阵元

用与得到重耦矩阵 $W_{[123]}{}^{k_2k_3}_{i_2i_3}$ 的矩阵元(42)类似的方法,可求得其他各重耦矩阵表式的矩阵元如下：

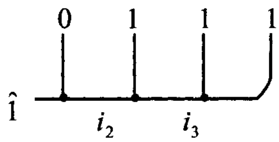
$$W_{[124]}{}^{k_2k_3}_{i_2i_3} = \begin{bmatrix} W_{[124]}{}^{10}_{10} & W_{[124]}{}^{12}_{10} & W_{[124]}{}^{32}_{10} \\ W_{[124]}{}^{10}_{12} & W_{[124]}{}^{12}_{12} & W_{[124]}{}^{32}_{12} \\ W_{[124]}{}^{10}_{32} & W_{[124]}{}^{12}_{32} & W_{[124]}{}^{32}_{32} \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3}\sqrt{3} & -\frac{\sqrt{6}}{6} \\ -\frac{2}{3}\sqrt{3} & 0 & -\sqrt{2}/2 \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}, \tag{43}$$

$$W_{[234]}{}^{k_2k_3}_{i_2i_3} = \begin{bmatrix} W_{[234]}{}^{10}_{10} & W_{[234]}{}^{12}_{10} & W_{[234]}{}^{32}_{10} \\ W_{[234]}{}^{10}_{12} & W_{[234]}{}^{12}_{12} & W_{[234]}{}^{32}_{12} \\ W_{[234]}{}^{10}_{32} & W_{[234]}{}^{12}_{32} & W_{[234]}{}^{32}_{32} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{44}$$

$$W_{[134]}{}^{k_2k_3}_{i_2i_3} = \begin{bmatrix} W_{[134]}{}^{10}_{10} & W_{[134]}{}^{12}_{10} & W_{[134]}{}^{32}_{10} \\ W_{[134]}{}^{10}_{12} & W_{[134]}{}^{12}_{12} & W_{[134]}{}^{32}_{12} \\ W_{[134]}{}^{10}_{32} & W_{[134]}{}^{12}_{32} & W_{[134]}{}^{32}_{32} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{2}{3}\sqrt{3} & -\frac{\sqrt{6}}{3} \\ \frac{2}{3}\sqrt{3} & 0 & 0 \\ \frac{\sqrt{6}}{3} & 0 & 0 \end{bmatrix}. \tag{45}$$

7.5 顶角$(\hat{1},0,1,1,1)$的重耦矩阵元

5 顶角$(\hat{1},0,1,1,1)$的展开图为


(46)

此时双重颜色 $i_2 i_3$ 应取值为 10 和 12. 类似地计算，将得到如下重耦矩阵元：

$$W_{[123]}{}^{k_2k_3}_{i_2i_3} = \begin{bmatrix} W_{[123]}{}^{10}_{10} & W_{[123]}{}^{12}_{10} \\ W_{[123]}{}^{10}_{12} & W_{[123]}{}^{12}_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad (47)$$

$$W_{[124]}{}^{k_2k_3}_{i_2i_3} = \begin{bmatrix} W_{[124]}{}^{10}_{10} & W_{[123]}{}^{12}_{10} \\ W_{[124]}{}^{10}_{12} & W_{[123]}{}^{12}_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad (48)$$

$$W_{[134]}{}^{k_2k_3}_{i_2i_3} = \begin{bmatrix} W_{[134]}{}^{10}_{10} & W_{[134]}{}^{12}_{10} \\ W_{[134]}{}^{10}_{12} & W_{[134]}{}^{12}_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad (49)$$

$$W_{[234]}{}^{k_2k_3}_{i_2i_3} = \begin{bmatrix} W_{[234]}{}^{10}_{10} & W_{[234]}{}^{12}_{10} \\ W_{[234]}{}^{10}_{12} & W_{[234]}{}^{12}_{12} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 0 \end{bmatrix}. \quad (50)$$

8. 体积算符 $\hat{V}$ 的期望值矩阵元及正规化子

如上的重耦矩阵元的计算，是为了最后得到体积算符 $\hat{V}$ 对顶角作用的期望值矩阵元，以便用于本文的计算. 将(42)—(45)式给出的重耦矩阵元代入(22)式，算符 $\hat{V}$ 对顶角$(\hat{1},2,1,1,1)$作用的期望值矩阵元，将被算出如下：

$$V^{k_2k_3}_{i_2i_3} = \begin{bmatrix} V^{10}_{10} & V^{12}_{10} & V^{32}_{10} \\ V^{10}_{12} & V^{12}_{12} & V^{32}_{12} \\ V^{10}_{32} & V^{12}_{32} & V^{32}_{32} \end{bmatrix}$$

$$= \frac{(\hbar k)^{3/2}}{4} \begin{bmatrix} 0 & \sqrt{\frac{5}{2}\sqrt{3}} & \sqrt{\frac{2}{3}\sqrt{6}} \\ \sqrt{\frac{5}{2}\sqrt{3}} & 0 & \sqrt[4]{2} \\ \sqrt{\frac{2}{3}\sqrt{6}} & \sqrt[4]{2} & 0 \end{bmatrix}. \quad (51)$$

对于顶角$(\hat{1},0,1,1,1)$，算符 $\hat{V}$ 对其作用的矩阵元为

$$V^{k_2k_3}_{i_2i_3} = \frac{(\hbar k)^{3/2}}{4} \begin{bmatrix} 0 & \sqrt{\frac{\sqrt{3}}{2}} \\ \sqrt{\frac{\sqrt{3}}{2}} & 0 \end{bmatrix}. \quad (52)$$

利用(30)式，顶角$(\hat{1},2,1,1,1)$的正规化子为

$$N_{10} = \frac{\sqrt{-6}}{6}, N_{12} = \frac{\sqrt{-2}}{3}, N_{32} = \frac{\sqrt{-1}}{2}. \quad (53)$$

顶角$(\hat{1},0,1,1,1)$的正规化子为

$$N_{10} = \frac{1}{2}, N_{12} = \frac{\sqrt{3}}{3}. \quad (54)$$

9. 度量矩阵算符对角分量的期望值

节 1 与节 2 证明了标架度量算符对用于进行 Gauss 编织的基 ψ_p 的作用为本征作用，求得了该作用的矩阵表示式. 下面将求出度量算符这种作用的期望值. 首先计算度量矩阵算符对角分量“00”($\alpha=\beta=0$)的期望值. 在本征方程(8a)中，因 $k=0$，有 $m=0$，故该分量可写成

$$M(S_0,S_0) = -\frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{s,t}\sum_{g=\pm1}\gamma_q(1)\gamma_g(s)\left(\frac{N_0}{N_0}\right)(1)$$
$$\times\left(\frac{N_{st}V^{st}_{10}}{N_{10}}\right)(\hat{1},1+q,1,1,1)\left(\frac{N_{10}V^{10}_{st}}{N_{st}}\right)(\hat{1},|s+g|,1,1,1)$$
$$\times\frac{\mathrm{Tet}\begin{bmatrix} s & 1 & 1+q \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & s & |s+g| \\ 1 & 1 & 2 \end{bmatrix}}{\Delta_1\theta(s,1,2)}, \quad (55)$$

此处由 3 顶角的相容条件知，$s=1$ 或 3. 当 $s=1$ 时，$t=0$ 或 2 当 $s=3$ 时，$t=2$. 利用 $\sum_{q=\pm1}$ 和 $\sum_{g=\pm1}$ 展开(55)式，得到

$$M(S_0,S_0) = -\frac{8}{(\hbar k)^2}\sum_{s,t}\left[\gamma_+(1)\gamma_+(s)\left(\frac{N_{st}V^{st}_{10}}{N_{10}}\right)(\hat{1},2,1,1,1)\right.$$

$$
\begin{aligned}
&\times \frac{N_{10}V_{st}^{10}}{N_{st}}(\hat{1},|s+1|,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} s & 1 & 2\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & s & |s+1|\\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(s,1,2)}\\
&+\gamma_+(1)\gamma_-(s)\frac{N_{st}V_{10}^{st}}{N_{10}}(\hat{1},2,1,1,1)\frac{N_{10}V_{st}^{10}}{N_{st}}(\hat{1},|s-1|,1,1,1)\\
&\times \frac{\mathrm{Tet}\begin{bmatrix} s & 1 & 2\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & s & |s-1|\\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(s,1,2)}+\gamma_-(1)\gamma_+(s)\\
&\times \frac{N_{st}V_{10}^{st}}{N_{10}}(\hat{1},0,1,1,1)\frac{N_{10}V_{st}^{10}}{N_{st}}(\hat{1},|s+1|,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} s & 1 & 0\\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(s,1,2)}\\
&\times \mathrm{Tet}\begin{bmatrix} 1 & s & |s+1|\\ 1 & 1 & 2\end{bmatrix}+\gamma_-(1)\gamma_-(s)\frac{N_{st}V_{10}^{st}}{N_{10}}(\hat{1},0,1,1,1)\\
&\times \left.\frac{N_{10}V_{st}^{10}}{N_{st}}(\hat{1},|s-1|,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} s & 1 & 0\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & s & |s-1|\\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(s,1,2)}\right]. \qquad (56)
\end{aligned}
$$

将(56)式的方括号中的 4 项分别记以 A，B，C 和 D，引入前面求得的顶角正规化子，以及用于顶角$(\hat{1},2,1,1,1)$和$(\hat{1},0,1,1,1)$的体积期望值矩阵元(由(51)和(52)式给出)，我们得到

$$
\begin{aligned}
A &= \gamma_+(1)\gamma_+(1)\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},2,1,1,1)\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},2,1,1,1)\\
&\quad\times \frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(1,1,2)}+\gamma_+(1)\gamma_+(1)\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},2,1,1,1)\\
&\quad\times \frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(2,1,2)}+\gamma_+(1)\gamma_+(3)\\
&\quad\times \frac{N_{32}V_{10}^{32}}{N_{10}}(\hat{1},2,1,1,1)\frac{N_{10}V_{32}^{10}}{N_{32}}(\hat{1},4,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} 3 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 3 & 4\\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(3,1,2)}\\
&= 0+\frac{\frac{\sqrt{-2}}{3}}{\frac{\sqrt{-6}}{6}}\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{5}{2}\sqrt{3}}\frac{\sqrt{-6}/6}{\sqrt{-2}/3}\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{5}{2}\sqrt{3}}\frac{\frac{3}{2}}{(-2)3}\frac{3}{2}+0\\
&= -\frac{15\sqrt{3}}{256}(\hbar k)^3,
\end{aligned}
$$

$$
\begin{aligned}
B &= \gamma_+(1)\gamma_+(1)\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},2,1,1,1)\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},0,1,1,1)\\
&\quad\times \frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(1,1,2)}+\gamma_+(1)\gamma_-(1)\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},2,1,1,1)\\
&\quad\times \frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(1,1,2)}+\gamma_+(1)\gamma_-(3)\\
&\quad\times \frac{N_{32}V_{10}^{32}}{N_{10}}(\hat{1},2,1,1,1)\frac{N_{10}V_{32}^{10}}{N_{32}}(\hat{1},2,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} 3 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 3 & 2\\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(3,1,2)}
\end{aligned}
$$

$$
\begin{aligned}
&= 0 + \left(-\frac{1}{2}\right)\frac{\frac{\sqrt{-2}}{3}\,\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{5}{2}\sqrt{3}}}{\sqrt{-6}/6}\,\frac{\frac{1}{2}\,\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{\sqrt{3}}{2}}}{\sqrt{3}/3}\,\frac{\frac{3}{2}3}{(-2)3} + \left(-\frac{3}{4}\right)\\
&\quad\times \frac{\frac{\sqrt{-1}}{2}\,\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{2}{3}\sqrt{6}}}{\sqrt{-6}/6}\,\frac{\frac{\sqrt{-6}}{6}\,\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{2}{3}\sqrt{6}}}{\sqrt{-1}/2}\,\frac{(-4)(-4)}{(-2)(-4)}\\
&= \frac{3\sqrt{15}-16\sqrt{6}}{256}(\hbar k)^3,
\end{aligned}
$$

$$
\begin{aligned}
C &= \gamma_-(1)\gamma_+(1)\,\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},0,1,1,1)\,\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},2,1,1,1)\\
&\quad\times\frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\Delta_1\theta(1,1,2)} + \gamma_-(1)\gamma_+(1)\,\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},0,1,1,1)\\
&\quad\times\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)\,\frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\Delta_1\theta(1,1,2)} + \gamma_-(1)\gamma_+(3)\\
&\quad\times\frac{N_{32}V_{10}^{32}}{N_{10}}(\hat{1},0,1,1,1)\,\frac{N_{10}V_{32}^{10}}{N_{32}}(\hat{1},4,1,1,1)\,\mathrm{Tet}\begin{bmatrix}3&1&0\\1&1&2\end{bmatrix}\frac{\mathrm{Tet}\begin{bmatrix}1&3&4\\1&1&2\end{bmatrix}}{\Delta_1\theta(3,1,2)}\\
&= 0 - \frac{1}{2}\,\frac{\frac{\sqrt{3}}{3}\,\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{\sqrt{3}}{2}}}{\frac{1}{2}}\,\frac{\frac{\sqrt{-6}}{6}\,\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{5}{2}\sqrt{3}}}{\sqrt{-2}/3}\,\frac{3\,\frac{3}{2}}{(-2)3} + 0\\
&= \frac{3\sqrt{15}}{256}(\hbar k)^3,
\end{aligned}
$$

$$
\begin{aligned}
D &= \gamma_-(1)\gamma_-(1)\,\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},0,1,1,1)\,\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},0,1,1,1)\\
&\quad\times\frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}}{\Delta_1\theta(1,1,2)} + \gamma_-(1)\gamma_-(1)\,\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},0,1,1,1)\\
&\quad\times\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\,\frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}}{\Delta_1\theta(1,1,2)} + \gamma_-(1)\gamma_-(3)\\
&\quad\times\frac{N_{32}V_{10}^{32}}{N_{10}}(\hat{1},0,1,1,1)\,\frac{N_{10}V_{32}^{10}}{N_{32}}(\hat{1},2,1,1,1)\,\frac{\mathrm{Tet}\begin{bmatrix}3&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&3&2\\1&1&2\end{bmatrix}}{\Delta_1\theta(3,1,2)}\\
&= 0 + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{\frac{\sqrt{3}}{3}\,\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{\sqrt{3}}{2}}}{\frac{1}{2}}\,\frac{\frac{1}{2}\,\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{\sqrt{3}}{2}}}{\sqrt{3}/3}\,\frac{3\cdot 3}{(-2)\cdot 3} + 0\\
&= \frac{3\sqrt{3}}{256}(\hbar k)^3 .
\end{aligned}
$$

将如上得到的 A，B，C 和 D 代入(56)式，可求得度量矩阵‘00’分量的期望值为

$$
\begin{aligned}
M(S_0,S_0) &= -\frac{8}{(\hbar k)^2}\left[\frac{-15\sqrt{3}+3\sqrt{15}-16\sqrt{6}+3\sqrt{15}-3\sqrt{3}}{256}\right]\hbar k\\
&= 1.473\,\hbar k . \qquad (57)
\end{aligned}
$$

其余对角分量的期望值，可类似地求得，它们是

$$M(s_1,s_1)=0.250\hbar k, M(s_2,s_2)=1.473\hbar k, M(s_3,s_3)=0.250\hbar k.$$

10. 度量矩阵算符非对角分量的期望值

再作为一个例子，首先给出“01”分量的期望值计算．由(16)式知，该分量的期望值为

$$\begin{aligned}M(s_0,s_1)=&\frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{g=\pm1}\sum_t\gamma_q(p)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p)\left\{\frac{N_{pt}V_{pk}^{pt}}{N_{pk}}(\hat{1},p+q,p,p,p)\right.\\&\times\left(\sum_\gamma\begin{Bmatrix}p&p&r\\p&p&t\end{Bmatrix}\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)(\hat{1},p+g,p,p,p)\right)\\&+\left(\sum_l\begin{Bmatrix}p&p&l\\p&p&k\end{Bmatrix}\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1},p+q,p,p,p)\right)\\&\left.\times\left(\sum_{l''}\begin{Bmatrix}s&p&l''\\p&p&t\end{Bmatrix}\sum_{l'}\begin{Bmatrix}p&p&l'\\p&s&l''\end{Bmatrix}\sum_l\begin{Bmatrix}p&p&l\\s&p&l'\end{Bmatrix}\left(\frac{N_{pm}V_{pl}^{pm}}{N_{pl}}\right)(\hat{1},p+g,p,p,p)\right)\right\}\\&\times\frac{\mathrm{Tet}\begin{bmatrix}2&p&p+q\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}p&p&p+g\\1&1&2\end{bmatrix}}{\theta(2,2,p)\theta(p,2,p)}.\end{aligned}\tag{58}$$

令(58)式中的两项分别为 M_{10} 和 M_{01}．对于 M_{10}，利用 $\sum\limits_{q=\pm1}$ 和 $\sum\limits_{g=\pm1}$ 将其展开，有

$$\begin{aligned}M_{10}=&\frac{8}{(\hbar k)^2}\sum_t\left[\gamma_+(1)\gamma_+(1)\frac{N_{1t}V_{10}^{1t}}{N_{10}}(\hat{1},2,1,1,1)\left(\sum_r\begin{Bmatrix}1&1&r\\1&1&t\end{Bmatrix}\frac{N_{10}V_{1r}^{10}}{N_{1r}}(\hat{1},2,1,1,1)\right)\right.\\&\times\frac{\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}+\gamma_+(1)\gamma_-(1)\frac{N_{1t}V_{10}^{1t}}{N_{10}}(\hat{1},2,1,1,1)\left(\sum_r\begin{Bmatrix}1&1&r\\1&1&t\end{Bmatrix}\right.\\&\left.\times\frac{N_{10}V_{1r}^{1t}}{N_{1r}}(\hat{1},0,1,1,1)\right)\frac{\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}+\gamma_-(1)\gamma_+(1)\\&\times\frac{N_{1t}V_{10}^{1t}}{N_{10}}(\hat{1},0,1,1,1)\left(\sum_r\begin{Bmatrix}1&1&r\\1&1&t\end{Bmatrix}\frac{N_{10}V_{pr}^{10}}{N_{pr}}(\hat{1},2,1,1,1)\right)\frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}\\&+\gamma_-(1)\gamma_-(1)\frac{N_{1t}V_{10}^{1t}}{N_{10}}(1,0,1,1,1)\left(\sum_r\begin{Bmatrix}1&1&r\\1&1&t\end{Bmatrix}\frac{N_{10}V_{1r}^{10}}{N_{1r}}(\hat{1},0,1,1,1)\right)\\&\left.\times\frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}\right].\end{aligned}\tag{59}$$

将(59)式的方括号中的 4 项分别记以 E,F,G 和 H．经分析知，颜色 t 只能取 0 或 2．经计算，可得如下结果：

$$\begin{aligned}E=&\gamma_+(1)\gamma_+(1)\frac{V_{12}V_{10}^{12}}{N_{10}}(\hat{1},2,1,1,1)\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\frac{V_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}\\=&\frac{5\sqrt{3}}{256}(\hbar k)^3,\end{aligned}$$

$$F=\gamma_+(1)\gamma_-(1)\frac{V_{12}V_{10}^{12}}{N_{10}}(\hat{1},2,1,1,1)\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\frac{V_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}$$

$$
= -\frac{2\sqrt{15}}{512}(\hbar k)^3,
$$

$$
G = \gamma_-(1)\gamma_+(1)\frac{V_{12}V_{10}^{12}}{N_{10}}(\hat{1},0,1,1,1)\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\frac{V_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}
$$

$$
= -\frac{2\sqrt{15}}{512}(\hbar k)^3,
$$

$$
E = \gamma_-(1)\gamma_-(1)\frac{V_{12}V_{10}^{12}}{N_{10}}(\hat{1},0,1,1,1)\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\frac{V_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}
$$

$$
= \frac{5\sqrt{3}}{256}(\hbar k)^3 .
$$

将 E，F，G 和 H 代入(59)式，可得

$$
M_{10} \doteq 0.083\hbar k .
$$

对于 M_{01}，经类似计算可得

$$
\begin{aligned}
M_{01} = {} & \frac{8}{(\hbar k)^2}\sum_{g=\pm 1}\sum_{t}\Bigg[\gamma_+(1)\gamma_+(1)\Bigg(\sum_{l}\begin{Bmatrix}1&1&l\\1&1&0\end{Bmatrix}\Bigg(\frac{N_{1t}V_{1l}^{1t}}{N_{1l}}(\hat{1},2,1,1,1)\Bigg)\Bigg) \\
& \times\Bigg(\sum_{r''}\begin{Bmatrix}1&1&r''\\1&1&t\end{Bmatrix}\sum_{r'}\begin{Bmatrix}1&1&r'\\1&1&r''\end{Bmatrix}\sum_{r}\begin{Bmatrix}1&1&r\\1&1&r'\end{Bmatrix}\frac{N_{10}V_{pr}^{10}}{N_{pr}}(\hat{1},2,1,1,1)\Bigg)\frac{\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2'\end{bmatrix}}{\theta(1,2,1)\theta(1,1,2)} \\
& + \gamma_+(1)\gamma_-(1)\Bigg(\sum_{l}\begin{Bmatrix}1&1&l\\1&1&0\end{Bmatrix}\Bigg(\frac{N_{1t}V_{1l}^{1t}}{N_{1l}}(\hat{1},2,1,1,1)\Bigg)\Bigg(\sum_{r''}\begin{Bmatrix}1&1&r''\\1&1&t\end{Bmatrix}\sum_{r'}\begin{Bmatrix}1&1&r'\\1&1&r''\end{Bmatrix}\sum_{r}\begin{Bmatrix}1&1&r\\1&1&r'\end{Bmatrix} \\
& \times\frac{N_{10}V_{1r}^{10}}{N_{1r}}(\hat{1},0,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\theta(1,2,1)}\frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}}{\theta(1,1,2)} + \gamma_-(1)\gamma_+(1)\Bigg(\sum_{l}\begin{Bmatrix}1&1&l\\1&1&0\end{Bmatrix}\frac{N_{1t}V_{1l}^{1t}}{N_{1l}}(\hat{1},0,1,1,1)\Bigg) \\
& \times\sum_{r''}\begin{Bmatrix}1&1&r''\\1&1&t\end{Bmatrix}\sum_{r'}\begin{Bmatrix}1&1&r'\\1&1&r''\end{Bmatrix}\sum_{r}\begin{Bmatrix}1&1&r\\1&1&r'\end{Bmatrix}\frac{N_{10}V_{1r}^{10}}{N_{1r}}(\hat{1},2,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix}s&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)} \\
& + \gamma_-(1)\gamma_-(1)\Bigg(\sum_{l}\begin{Bmatrix}1&1&l\\1&1&0\end{Bmatrix}\frac{N_{1t}V_{1l}^{1t}}{N_{1l}}(\hat{1},0,1,1,1)\Bigg)\Bigg(\sum_{r''}\begin{Bmatrix}1&1&r''\\1&1&t\end{Bmatrix}\sum_{r'}\begin{Bmatrix}1&1&r'\\1&1&r''\end{Bmatrix}\sum_{r}\begin{Bmatrix}1&1&r\\1&1&r'\end{Bmatrix} \\
& \times\frac{N_{10}V_{1r}^{10}}{N_{1r}}(\hat{1},0,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}\Bigg].
\end{aligned}
\tag{60}
$$

令上式方括号中的 4 项分别为 E'，F'，G' 和 H'，经计算可得

$$
\begin{aligned}
E' = {} & \begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\Bigg(\begin{Bmatrix}1&1&0\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&0\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1) \\
& + \begin{Bmatrix}1&1&0\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1) + \begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&0\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix} \\
& \times\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1) + \begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)\Bigg)\frac{\frac{3}{2}\cdot\frac{3}{2}}{3\cdot 3} \\
& + \begin{Bmatrix}1&1&0\\1&1&0\end{Bmatrix}\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},2,1,1,1)\Bigg(\begin{Bmatrix}1&1&0\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&0\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix} + \begin{Bmatrix}1&1&0\\1&1&2\end{Bmatrix}
\end{aligned}
$$

$$\times\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}+\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&0\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}+\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\Bigg)$$

$$\times\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)\ \frac{\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\theta(1,1,2)\,\theta(1,1,2)}$$

$$=\frac{10\sqrt{3}}{512}(\hbar k)^3,$$

$$F'=\gamma_+(1)\gamma_-(1)\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)\Bigg(\begin{Bmatrix}1&1&0\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&0\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}$$

$$+\begin{Bmatrix}1&1&0\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}+\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&0\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}+\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}$$

$$\times\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\Bigg)\ \frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\ \frac{\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}}{\theta(1,1,2)\,\theta(1,1,2)}$$

$$+\gamma_+(1)\gamma_-(1)\begin{Bmatrix}1&1&0\\1&1&0\end{Bmatrix}\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},2,1,1,1)\Bigg(\begin{Bmatrix}1&1&0\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&0\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}$$

$$+\begin{Bmatrix}1&1&0\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}+\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&0\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}$$

$$+\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\ \frac{\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}}{\theta(1,1,2)\,\theta(1,1,2)}$$

$$=-\frac{\sqrt{15}}{256}(\hbar k)^3,$$

$$G'=\gamma_-(1)\gamma_+(1)\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\ \frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)$$

$$\times\frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\theta(1,1,2)\,\theta(1,1,2)}\gamma_-(1)\gamma_+(1)\begin{Bmatrix}1&1&0\\1&1&0\end{Bmatrix}\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)$$

$$\times\frac{4}{8}\ \frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)\ \frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\theta(1,1,2)\,\theta(1,1,2)}$$

$$=-\frac{\sqrt{15}}{256}(\hbar k)^3,$$

$$H'=\gamma_-(1)\gamma_-(1)\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\ \frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)$$

$$\times\frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}}{\theta(1,1,2)\,\theta(1,1,2)}+\gamma_-(1)\gamma_-(1)\begin{Bmatrix}1&1&0\\1&1&0\end{Bmatrix}\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},0,1,1,1)\ \frac{4}{8}$$

$$\times\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\ \frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}}{\theta(1,1,2)\,\theta(1,1,2)}$$

$$=\frac{\sqrt{3}}{256}(\hbar k)^3.$$

由 E'，F'，G'和 H'，可得

$$M_{01} \doteq 0.083\hbar k.$$

将 M_{10}与 M_{01}结合，可得

$$M(s_0, s_1) = M_{10} + M_{01} = 0.165\hbar k. \quad (61)$$

类似地可得其余非对角分量为

$$M(s_0, s_2) = 0.330\hbar k,$$
$$M(s_0, s_3) = 0.165\hbar k,$$
$$M(s_1, s_2) = 0.165\hbar k,$$
$$M(s_1, s_3) = -0.165\hbar k,$$
$$M(s_2, s_3) = 0.165\hbar k.$$

11. Gauss 编织态的自旋几何

我们已经得到了度量矩阵算符 $\hat{M}(s_\alpha, s_\beta)$作用于态 $\psi_p(p=1)$所携带的顶角 ϕ_k 上的期望值. 现在根据 Penrose 的自旋定理[11]，探讨 Gauss 编织态的自旋几何.

11.1. $k = 0$ 的情况

顶角 υ 处毗邻的腿的相位切方向的夹角定义为

$$\theta(s_\alpha, s_\beta) = \cos^{-1}\left(M(s_\alpha, s_\beta)/\sqrt{M(s_\alpha, s_\alpha)}\sqrt{M(s_\beta, s_\beta)}\right). \quad (62)$$

将前面 $k = 0$ 时的度规算符的期望值写成矩阵形式，有

$$M(s_\alpha, s_\beta) = \hbar k\begin{bmatrix} 1.473 & 0.165 & 0.330 & 0.165 \\ 0.165 & 0.250 & 0.165 & -0.165 \\ 0.330 & 0.165 & 1.473 & 0.165 \\ 0.165 & -0.165 & 0.165 & 0.250 \end{bmatrix}. \quad (63)$$

将(63)式中的相应值代入(62)式，得到空间离散状态下的顶角 υ 处的 4 个切方向间的 6 个夹角为

$$\theta(s_0, s_1) = \cos^{-1}0.274 = 74.12°,$$
$$\theta(s_0, s_2) = \cos^{-1}0.224 = 77.05°,$$
$$\theta(s_0, s_3) = \cos^{-1}0.274 = 74.12°,$$
$$\theta(s_1, s_2) = \cos^{-1}0.274 = 74.12°,$$
$$\theta(s_1, s_3) = \cos^{-1}(-0.656) = 130.99°,$$
$$\theta(s_2, s_3) = \cos^{-1}0.274 = 74.12°.$$

顶角 υ 处切矢量的长度，经计算分别为

$$|\dot{s}_0| = 1.214(\hbar k)^{1/2}, \quad |\dot{s}_1| = 0.5(\hbar k)^{1/2},$$
$$|\dot{s}_2| = 1.214(\hbar k)^{1/2}, \quad |\dot{s}_3| = 0.5(\hbar k)^{1/2};$$

切矢量的平均长度为

$$\frac{1}{4}\sum_\alpha\sqrt{M(s_\alpha, s_\alpha)} = 0.875(\hbar k)^{1/2}.$$

这一平均长度可用于为不同顶角间分配间隙.

11.2. $k = 2$ 的情况

依照前面的作法，但取 $k = 2$，继续完成相应的计算，可得 $k = 2$ 时的度量矩阵算符 $\hat{M}(s_\alpha, s_\beta)$的期望值矩阵为

$$M(s_\alpha, s_\beta) = \hbar k\begin{bmatrix} 1.309 & 0 & 0.330 & 0 \\ 0 & 1.642 & 0 & 0.165 \\ 0.330 & 0 & 1.309 & 0 \\ 0 & 0.165 & 0 & 1.642 \end{bmatrix},$$

Gauss 编织态的基的顶角 υ 处腿的相位切方向间的夹角为

$$\theta(s_0, s_1) = \cos^{-1}0 = 90°,$$
$$\theta(s_0, s_2) = \cos^{-1}0.165 = 80.5°,$$
$$\theta(s_0, s_3) = \cos^{-1}0 = 90°,$$
$$\theta(s_1, s_2) = \cos^{-1}0 = 90°,$$
$$\theta(s_1, s_3) = \cos^{-1}0.101 = 84.2°,$$
$$\theta(s_2, s_3) = \cos^{-1}0 = 90°,$$

切矢量的长度为

$$|\dot{S}_0| = 1.144(\hbar k)^{1/2},$$
$$|\dot{S}_1| = 1.281(\hbar k)^{1/2},$$
$$|\dot{S}_2| = 1.144(\hbar k)^{1/2},$$
$$|\dot{S}_3| = 1.281(\hbar k)^{1/2},$$

切矢量的平均长度为

$$\frac{1}{4}\sum_\alpha\sqrt{M(s_\alpha, s_\alpha)} = 1.213(\hbar k)^{1/2}.$$

本文得到的关于度量矩阵、切方向间的夹角以及切矢量的长度的结果，在 Gauss 编织态的腿指标 0 ↔2 和 1↔3 的同时置换下全部是对称的. 并且发现，度量矩阵算符对角分量的期望值是纯正定的，并且求得的切矢量长度也全是正的. 这些性质说明，标架度量算符定义在 Gauss 编织态上是可行的，得到的由 4 个非迷向切矢量构成的几何顶角，对于空间编织的进一步研究将有重要应用价值，将对空间由量子化离散状态向半经典的编织状态过度，提供原理上和技巧上的依据.

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Eigenaction of metric operator on Gaussian weave state and spin-geometry*

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Abstract

In the recouping theorem and the graph calculation for loop quantum gravity, it is proved that the action of metric matrix operator on Gaussian weave state is an eigenaction, and the representation matrix elements of the metric operator and their expectation values are calculated. The values of the length of tangent vectors with 4 edges ($P=1$) adjacent to the vertex of Gaussian weave state ψ_p, as well as the angles between them, are also obtained in the cases of $k=0$ and $k=2$.

Keywords: representation matrix of metric operator, expectation values of metric operator, angles between tangential directions, lengths of tangent vectors

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