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离散差分序列变质量力学系统的 Mei 对称性

黄晓虹[1)] 张晓波[2)] 施沈阳[2)†]

1)(温州大学物理与电子信息学院,温州 325000)

2)(浙江理工大学物理系,杭州 310018)

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研究离散差分序列变质量力学系统的 Mei 对称性与守恒量.定义离散系统的差分序列方程在无限小变换群下的形式不变性为 Mei 对称性.给出由 Mei 对称性得到守恒量的判据.举例说明结果的应用.

关键词:离散力学,变质量系统,Mei 对称性,离散守恒量

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1.引 言

对称性是物理、约束力学、数学物理等领域的重要研究议题.对称性理论有许多用途,如常微分方程的积分,偏微分方程的约化、线性化和渐进解以及分岔和控制等,其中一个比较重要的用途就是用来寻求守恒量.对称性与守恒量存在潜在关系,守恒量不仅具有重要的数学意义,而且反映出深刻的物理本质.近来,运用无限小变换群的方法研究约束力学系统的对称性和守恒量取得了一系列有意义的结果[1—10].离散约束力学系统的对称性和守恒量是近十年来分析力学领域的研究方向之一[11—16],主要思想是利用力学的变分原理来构建系统的离散差分运动方程,这种离散序列方程提供了一种递推算法,而这种算法通常保留了连续系统的辛结构等几何结构和对称性、可积性等方程性质.因此,在讨论约束力学系统的性质、运动规划、系统控制、数值仿真计算时,这些离散算法通常比传统的算法有更好的计算效率、保结构和不变性等优点.

Lie 对称性和 Noether 对称性是近代寻求连续约束力学系统守恒量的两种常用的重要方法.Lie 对称性考察运动微分方程在无限小变换群下的不变性[1,2,17—22].Noether 对称性是基于 Hamilton 泛函作用量在无限小变换群下的不变性[1,2,23,24].Mei 对称性是梅凤翔提出的一类不同于 Noether 对称性和 Lie 对称性的对称性形式,这种新对称性理论的主要思想是考虑力学系统的动力学函数在无限小 Lie 群变换下满足原来运动微分方程形式的一种动力学不变性,动力学函数包括 Lagrange 函数、Hamilton 函数、非势广义力、各类完整的与非完整的约束力、各类形式完整的与非完整的约束方程、各类非保守力等等,这些动力学函数在无限小 Lie 群下做动力学变换后变换为新的动力学函数,新的动力学函数满足原来系统微分方程的形式保持不变,系统的各类方程包括 Euler-Lagrange 方程、Hamilton 正则方程、各类约束方程等等.Mei 对称性更主要是侧重于力学系统的动力学不变性,自 2000 年提出以来,成为研究约束力学系统守恒量的有力工具之一,得到了许多有意义的运用和推广[25—31].

Lie 对称性和 Noether 对称性方法近来被推广运用到离散系统,其中,Levi 等人[32—34]从数学方面较详细研究了离散差分方程和微分差分方程的 Lie 对称性理论,Dorodnitsyn[35]建立了离散 Lagrange 系统的 Noether 理论.而 Mei 对称性这一新的对称性形式还未应用于离散力学系统.本文在连续约束系统的 Mei 对称性与守恒量研究的基础上进一步讨论离散系统的 Mei 对称性,研究了离散差分序列变质量系统的 Mei 对称性,并给出由 Mei 对称性得到离散形式守恒量的条件.

2.离散变质量系统的 Mei 对称性与守恒量

考虑一个理想完整约束力学系统的位形由 n

† 通讯联系人.E-mail:hzshishenyang@sina.com

个广义坐标 $q_s(s=1,\cdots,n)$确定，并设质量的变化为 $m=m(t,q_s,\dot{q}_s)$，系统的 Lagrange 函数，非势广义力和广义反推力分别表示为 $L(t,q_s,\dot{q}_s)$，$Q_s(t,q_s,\dot{q}_s)$和 $P_s(t,q_s,\dot{q}_s)$．在离散力学中，用离散坐标的序列 $q_{s,i}(t_i)(i=0,1,\cdots,N)$来替代连续的位形坐标 $q_s(s=1,\cdots,n)$，相应离散形式的质量、Lagrange 函数、非势广义力和广义反推力为 $m_d(t_k,t_{k+1},q_{s,k},q_{s,k+1})$，$L_d(t_k,t_{k+1},q_{s,k},q_{s,k+1})$，$Q_{s,d}(t_k,t_{k+1},q_{s,k},q_{s,k+1})$，$P_{s,d}(t_k,t_{k+1},q_{s,k},q_{s,k+1})$，$(k=0,1,\cdots,N-1)$．系统的离散能量演化方程和离散 Euler-Lagrange 方程有形式[36]

$$\begin{aligned}&D_1L_d(\phi_{s,k})(t_{k+1}-t_k)\\&+D_2L_d(\phi_{s,k-1})(t_k-t_{k-1})\\&+L_d(\phi_{s,k-1})-L_d(\phi_{s,k})\\&-[Q_{s,d}(\phi_{s,k})+P_{s,d}(\phi_{s,k})](q_{s,k+1}-q_{s,k})=0,\end{aligned}\tag{1}$$

$$\begin{aligned}&D_3L_d(\phi_{s,k})(t_{k+1}-t_k)\\&+D_4L_d(\phi_{s,k-1})(t_k-t_{k-1})\\&+[Q_{s,d}(\phi_{s,k})+P_{s,d}(\phi_{s,k})](t_{k+1}-t_k)=0,\end{aligned}\tag{2}$$

其中，D_j 表示对离散函数第 j 个变量的偏导数，$\phi_{s,k}=(t_k,t_{k+1},q_{s,k},q_{s,k+1})$表示离散序列．

取离散时间和广义坐标的无限小变换群

$$t_k^*=t_k+\Delta t_k=t_k+\varepsilon\tau_k(t_k,q_{s,k}),\tag{3}$$

$$q_{s,k}^*=q_{s,k}+\Delta q_{s,k}=q_{s,k}+\varepsilon\xi_{s,k}(t_k,q_{s,k}),\tag{4}$$

其中 τ_k，$\xi_{s,k}$是离散生成元函数，生成元函数的矢量场依赖于一个离散点，表示为

$$X_d^{(0)}=\tau_k\frac{\partial}{\partial t_k}+\xi_{s,k}\frac{\partial}{\partial q_{s,k}},\tag{5}$$

其一次扩展依赖于相邻的两个离散点，表示为

$$X_d^{(1)}=X_d^{(0)}+\tau_{k+1}\frac{\partial}{\partial t_{k+1}}+\xi_{s,k+1}\frac{\partial}{\partial q_{s,k+1}}.\tag{6}$$

取离散变量和离散函数的递推算符和一次导数算符

$$R_\pm f(z_{s,k})=f(z_{s,k\pm1}),\tag{7}$$

$$D_df(z_s,k)=\frac{R_+f(z_{s,k})-f(z_{s,k})}{t_{k+1}-t_k}.\tag{8}$$

利用变换(3)，(4)，(6)和算符(7)，(8)，动力学函数 $L_d(\phi_{s,k})$，$L_d(\phi_{s,k-1})$，$Q_{s,d}(\phi_{s,k})$，$P_{s,d}(\phi_{s,k})$在无限小变换群下的动力学变换为

$$\begin{aligned}L_d(\phi_{s,k}^*)&=L_d(t_k^*,t_{k+1}^*,q_{s,k}^*,q_{s,k+1}^*)\\&=L_d(\phi_{s,k})+\varepsilon X_d^{(1)}[L_d(\phi_{s,k})]\\&\quad+o(\varepsilon^2)+\cdots,\end{aligned}\tag{9}$$

$$\begin{aligned}L_d(\phi_{s,k-1}^*)&=L_d(t_{k-1}^*,t_k^*,q_{s,k-1}^*,q_{s,k}^*)\\&=L_d(\phi_{s,k-1})+\varepsilon R_-X_d^{(1)}[L_d(\phi_{s,k-1})]\\&\quad+o(\varepsilon^2)+\cdots,\end{aligned}\tag{10}$$

$$\begin{aligned}Q_{s,d}(\phi_{s,k}^*)&=Q_{s,d}(t_k^*,t_{k+1}^*,q_{s,k}^*,q_{s,k+1}^*)\\&=Q_{s,d}(\phi_{s,k})+\varepsilon X_d^{(1)}[Q_{s,d}(\phi_{s,k})]\\&\quad+o(\varepsilon^2)+\cdots,\end{aligned}\tag{11}$$

$$\begin{aligned}P_{s,d}(\phi_{s,k}^*)&=P_{s,d}(t_k^*,t_{k+1}^*,q_{s,k}^*,q_{s,k+1}^*)\\&=P_{s,d}(\phi_{s,k})+\varepsilon X_d^{(1)}[P_{s,d}(\phi_{s,k})]\\&\quad+o(\varepsilon^2)+\cdots,\end{aligned}\tag{12}$$

其中，$o(\varepsilon^2)+\cdots$表示二阶及高阶的无限小量，在(10)式中有算符关系式

$$\begin{aligned}R_-X_d^{(1)}&=\tau_{k-1}\frac{\partial}{\partial t_{k-1}}+\tau_k\frac{\partial}{\partial t_k}\\&\quad+\xi_{s,k-1}\frac{\partial}{\partial q_{s,k-1}}+\xi_{s,k}\frac{\partial}{\partial q_{s,k}}.\end{aligned}\tag{13}$$

动力学变换(9)—(12)代入离散方程(1)和(2)得到

$$\begin{aligned}&D_1L_d(\phi_{s,k}^*)(t_{k+1}-t_k)\\&+D_2L_d(\phi_{s,k-1}^*)(t_k-t_{k-1})\\&+L_d(\phi_{s,k-1}^*)-L_d(\phi_{s,k}^*)\\&-[Q_{s,d}(\phi_{s,k}^*)+P_{s,d}(\phi_{s,k}^*)](q_{s,k+1}-q_{s,k})\\=&D_1\{L_d(\phi_{s,k})+\varepsilon X_d^{(1)}[L_d(\phi_{s,k})]\\&+o(\varepsilon^2)+\cdots\}(t_{k+1}-t_k)\\&+D_2\{L_d(\phi_{s,k-1})+\varepsilon R_-X_d^{(1)}[L_d(\phi_{s,k-1})]\\&+o(\varepsilon^2)+\cdots\}(t_k-t_{k-1})\\&+\{L_d(\phi_{s,k-1})+\varepsilon R_-X_d^{(1)}[L_d(\phi_{s,k-1})]\\&+o(\varepsilon^2)+\cdots\}\\&-\{L_d(\phi_{s,k})+\varepsilon X_d^{(1)}[L_d(\phi_{s,k})]\\&+o(\varepsilon^2)+\cdots\}\\&-\{Q_{s,d}(\phi_{s,k})+\varepsilon X_d^{(1)}[Q_{s,d}(\phi_{s,k})]\\&+o(\varepsilon^2)+\cdots\}(q_{s,k+1}-q_{s,k})\\&-\{P_{s,d}(\phi_{s,k})+\varepsilon X_d^{(1)}[P_{s,d}(\phi_{s,k})]\\&+o(\varepsilon^2)+\cdots\}(q_{s,k+1}-q_{s,k})=0,\end{aligned}\tag{14}$$

$$\begin{aligned}&D_3L_d(\phi_{s,k}^*)(t_{k+1}-t_k)\\&+D_4L_d(\phi_{s,k-1}^*)(t_k-t_{k-1})\\&+[Q_{s,d}(\phi_{s,k}^*)+P_{s,d}(\phi_{s,k}^*)](t_{k+1}-t_k)\\=&D_3\{L_d(\phi_{s,k})+\varepsilon X_d^{(1)}[L_d(\phi_{s,k})]\\&+o(\varepsilon^2)+\cdots\}(t_{k+1}-t_k)\\&+D_4\{L_d(\phi_{s,k-1})+\varepsilon R_-X_d^{(1)}[L_d(\phi_{s,k-1})]\\&+o(\varepsilon^2)+\cdots\}(t_k-t_{k-1})\\&+\{Q_{s,d}(\phi_{s,k})+\varepsilon X_d^{(1)}[Q_{s,d}(\phi_{s,k})]\end{aligned}$$

$$+ o(\varepsilon^2) + \cdots\}(t_{k+1} - t_k)$$
$$+ \{P_{s,\mathrm{d}}(\phi_{s,k}) + \varepsilon X_\mathrm{d}^{(1)}[P_{s,\mathrm{d}}(\phi_{s,k})]$$
$$+ o(\varepsilon^2) + \cdots\}(t_{k+1} - t_k) = 0. \quad (15)$$

略去二阶及高阶小量 $o(\varepsilon^2)+\cdots$，结合(1)和(2)式，并有离散方程(1)和(2)的形式不变，可得到

$$D_1\{X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})]\}(t_{k+1} - t_k)$$
$$+ D_2\{R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]\}(t_k - t_{k-1})$$
$$+ R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]$$
$$- X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})] - \{X_\mathrm{d}^{(1)}[Q_{s,\mathrm{d}}(\phi_{s,k})$$
$$+ P_{s,\mathrm{d}}(\phi_{s,k})]\}(q_{s,k+1} - q_{s,k}) = 0, \quad (16)$$

$$D_3\{X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})]\}(t_{k+1} - t_k)$$
$$+ D_4\{R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]\}(t_k - t_{k-1})$$
$$- \{X_\mathrm{d}^{(1)}[Q_{s,\mathrm{d}}(\phi_{s,k}) + P_{s,\mathrm{d}}(\phi_{s,k})]\}(t_{k+1} - t_k) = 0. \quad (17)$$

判据 1 如果生成元 τ_k，$\xi_{s,k}$ 满足方程(16)和(17)，则相应的离散能量演化方程(1)和运动方程(2)的形式不变性为离散差分序列变质量系统的 Mei 对称性，方程(16)和(17)称为 Mei 对称性的确定方程.

判据 2 如果生成元 τ_k，$\xi_{s,k}$ 满足方程(16)和(17)，且存在离散规范函数 $G_{Mk}(t_k, q_{s,k})$ 使下列等式成立

$$X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})]D_\mathrm{d}(\tau_k) + X_\mathrm{d}^{(1)}\{X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})]\}$$
$$+ X_\mathrm{d}^{(1)}[Q_{s,\mathrm{d}}(\phi_{s,k}) + P_{s,\mathrm{d}}(\phi_{s,k})]$$
$$\times [\xi_{s,k} - \tau_k D_\mathrm{d}(q_{s,k})] + D_\mathrm{d}(G_{Mk}) = 0, \quad (18)$$

则离散差分序列变质量系统存在如下离散守恒量

$$\tau_k R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]$$
$$+ \tau_k(t_k - t_{k-1})D_2 R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]$$
$$+ \xi_{s,k}(t_k - t_{k-1})D_4 R\_X_\mathrm{d}^{(1)}$$
$$\times [L_\mathrm{d}(\phi_{s,k-1})] + G_{Mk} = \mathrm{const}. \quad (19)$$

(18)式称为 Mei 等式.

证明 改写(16)和(17)式，有

$$D_1\{X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})]\}$$
$$+ \frac{t_k - t_{k-1}}{t_{k+1} - t_k}D_2\{R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]\}$$
$$+ \frac{R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})] - X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})]}{t_{k+1} - t_k}$$
$$- X_\mathrm{d}^{(1)}[Q_{s,\mathrm{d}}(\phi_{s,k}) + P_{s,\mathrm{d}}(\phi_{s,k})]D_\mathrm{d}(q_{s,k}) = 0, \quad (20)$$

$$D_3\{X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})]\}$$
$$+ \frac{t_k - t_{k-1}}{t_{k+1} - t_k}D_4\{R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]\}$$
$$+ X_\mathrm{d}^{(1)}[Q_{s,\mathrm{d}}(\phi_{s,k}) + P_{s,\mathrm{d}}(\phi_{s,k})] = 0, \quad (21)$$

展开等式(18)，得到

$$X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})]D_\mathrm{d}(\tau_k)$$
$$+ X_\mathrm{d}^{(1)}\{X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})]\}$$
$$+ X_\mathrm{d}^{(1)}[Q_{s,\mathrm{d}}(\phi_{s,k}) + P_{s,\mathrm{d}}(\phi_{s,k})]$$
$$\times [\xi_{s,k} - \tau_k D_\mathrm{d}(q_{s,k})] + D_\mathrm{d}(G_{Mk})$$
$$= \tau_k\Big\{D_1\{X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})]\}$$
$$+ \frac{t_k - t_{k-1}}{t_{k+1} - t_k}D_2\{R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]\}$$
$$+ \frac{R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})] - X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})]}{t_{k+1} - t_k}$$
$$- X_\mathrm{d}^{(1)}[Q_{s,\mathrm{d}}(\phi_{s,k}) + P_{s,\mathrm{d}}(\phi_{s,k})]D_\mathrm{d}(q_{s,k})\Big\}$$
$$+ \xi_{s,k}\Big\{D_3\{X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})]\}$$
$$+ \frac{t_k - t_{k-1}}{t_{k+1} - t_k}D_4\{R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]\}$$
$$+ X_\mathrm{d}^{(1)}[Q_{s,\mathrm{d}}(\phi_{s,k}) + P_{s,\mathrm{d}}(\phi_{s,k})]\Big\}$$
$$+ D_\mathrm{d}\{\tau_k R\_X_\mathrm{d}^{(a)}[L_\mathrm{d}(\phi_{s,k-1})]$$
$$+ \tau_k(t_k - t_{k-1})D_2 R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]$$
$$+ \xi_{s,k}(t_k - t_{k-1})D_4 R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]$$
$$+ G_{Mk}\} = 0, \quad (22)$$

在(22)式中利用了递推算符关系式

$$R_+ R\_X_\mathrm{d}^{(1)}[L_d(\phi_{s,k-1})] = X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})], \quad (23)$$

结合(20)和(21)式，(22)式变为

$$D_\mathrm{d}\{\tau_k R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]$$
$$+ \tau_k(t_k - t_{k-1})D_2 R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]$$
$$+ \xi_{s,k}(t_k - t_{k-1})D_4 R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]$$
$$+ G_{Mk}\} = 0, \quad (24)$$

则由(24)式得到离散 Mei 守恒量

$$\tau_k R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]$$
$$+ \tau_k(t_k - t_{k-1})D_2 R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]$$
$$+ \xi_{s,k}(t_k - t_{k-1})D_4 R\_X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k-1})]$$
$$+ G_{Mk} = \mathrm{const}. \quad (25)$$

3. 算 例

考虑一个变质量系统

$$L = m(\dot{q}_1^2 + \dot{q}_2^2), \quad (26)$$

$$Q_s = \alpha m(\dot{q}_1^2 + \dot{q}_2^2), P_s = \beta m(\dot{q}_1^2 + \dot{q}_2^2), \quad (27)$$

$$m = m_0 \exp\left(-\frac{1}{2}\gamma t\right), \tag{28}$$

其中 $\alpha, \beta, \gamma, m_0$ 为常系数，研究其离散系统的 Mei 对称性与守恒量.

由离散差分序列变换得到

$$L_{\mathrm{d}}(\phi_{s,k}) = m_{\mathrm{d}}\left[\left(\frac{q_{1,k+1}-q_{1,k}}{t_{t+1}-t_k}\right)^2 + \left(\frac{q_{2,k+1}-q_{2,k}}{t_{k+1}-t_k}\right)^2\right], \tag{29}$$

$$Q_{s,\mathrm{d}}(\phi_{s,k}) = \alpha m_{\mathrm{d}}\left(\frac{q_{1,k+1}-q_{1,k}}{t_{t+1}-t_k} + \frac{q_{2,k+1}-q_{2,k}}{t_{k+1}-t_k}\right), \tag{30}$$

$$P_{s,\mathrm{d}}(\phi_{s,k}) = \beta m_{\mathrm{d}}\left(\frac{q_{1,k+1}-q_{1,k}}{t_{t+1}-t_k} + \frac{q_{2,k+1}-q_{2,k}}{t_{k+1}-t_k}\right), \tag{31}$$

$$m_{\mathrm{d}} = m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right], \tag{32}$$

分别计算 $D_j[L_{\mathrm{d}}(\phi_{s,k})]$，$D_j[Q_{s,\mathrm{d}}(\phi_{s,k})]$，$D_j[P_{s,\mathrm{d}}(\phi_{s,k})]$，$(j=1,2,3,4)$

$$\begin{aligned} D_1[L_{\mathrm{d}}(\phi_{s,k})] &= -\frac{1}{2}\gamma m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \times\left[\left(\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\right)^2 + \left(\frac{q_{2,k+1}-q_{2,k}}{t_{k+1}-t_k}\right)^2\right] \\ &+ 2m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \times\left[\frac{(q_{1,k+1}-q_{1,k})^2}{(t_{k+1}-t_k)^3} + \frac{(q_{2,k+1}-q_{2,k})^2}{(t_{k+1}-t_k)^3}\right], \end{aligned} \tag{33}$$

$$\begin{aligned} D_2[L_{\mathrm{d}}(\phi_{s,k})] &= -\frac{1}{2}\gamma m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \times\left[\left(\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\right)^2 + \left(\frac{q_{2,k+1}-q_{2,k}}{t_{k+1}-t_k}\right)^2\right] \\ &- 2m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \times\left[\frac{(q_{1,k+1}-q_{1,k})^2}{(t_{k+1}-t_k)^3} + \frac{(q_{2,k+1}-q_{2,k})^2}{(t_{k+1}-t_k)^3}\right], \end{aligned} \tag{34}$$

$$D_3[L_{\mathrm{d}}(\phi_{s,k})] = -2m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \times\left[\frac{q_{1,k+1}-q_{1,k}}{(t_{k+1}-t_k)^2} + \frac{q_{2,k+1}-q_{2,k}}{(t_{k+1}-t_k)^2}\right], \tag{35}$$

$$D_4[L_{\mathrm{d}}(\phi_{s,k})] = 2m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \times\left[\frac{q_{1,k+1}-q_{1,k}}{(t_{k+1}-t_k)^2} + \frac{q_{2,k+1}-q_{2,k}}{(t_{k+1}-t_k)^2}\right], \tag{36}$$

$$\begin{aligned} D_1[Q_{s,\mathrm{d}}(\phi_{s,k})] &= -\frac{1}{2}\gamma\alpha m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \times\left(\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k} + \frac{q_{2,k+1}-q_{2,k}}{t_{k+1}-t_k}\right) \\ &+ m_0\alpha \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \times\left[\frac{q_{1,k+1}-q_{1,k}}{(t_{k+1}-t_k)^2} + \frac{q_{2,k+1}-q_{2,k}}{(t_{k+1}-t_k)^2}\right], \end{aligned} \tag{37}$$

$$\begin{aligned} D_2[Q_{s,\mathrm{d}}(\phi_{s,k})] &= -\frac{1}{2}\gamma\alpha m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \times\left(\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k} + \frac{q_{2,k+1}-q_{2,k}}{t_{k+1}-t_k}\right) \\ &- m_0\alpha \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \times\left[\frac{q_{1,k+1}-q_{1,k}}{(t_{k+1}-t_k)^2} + \frac{q_{2,k+1}-q_{2,k}}{(t_{k+1}-t_k)^2}\right], \end{aligned} \tag{38}$$

$$D_3[Q_{s,\mathrm{d}}(\phi_{s,k})] = \frac{-2\alpha m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right]}{t_{k+1}-t_k}, \tag{39}$$

$$D_4[Q_{s,\mathrm{d}}(\phi_{s,k})] = \frac{2\alpha m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right]}{t_{k+1}-t_k}, \tag{40}$$

$$\begin{aligned} D_1[P_{s,\mathrm{d}}(\phi_{s,k})] &= -\frac{1}{2}\gamma\beta m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \times\left(\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k} + \frac{q_{2,k+1}-q_{2,k}}{t_{k+1}-t_k}\right) \\ &+ m_0\beta \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \times\left[\frac{q_{1,k+1}-q_{1,k}}{(t_{k+1}-t_k)^2} + \frac{q_{2,k+1}-q_{2,k}}{(t_{k+1}-t_k)^2}\right], \end{aligned} \tag{41}$$

$$\begin{aligned} D_2[P_{s,\mathrm{d}}(\phi_{s,k})] &= -\frac{1}{2}\gamma\beta m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \times\left(\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k} + \frac{q_{2,k+1}-q_{2,k}}{t_{k+1}-t_k}\right) \\ &- m_0\beta \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \times\left[\frac{q_{1,k+1}-q_{1,k}}{(t_{k+1}-t_k)^2} + \frac{q_{2,k+1}-q_{2,k}}{(t_{k+1}-t_k)^2}\right], \end{aligned} \tag{42}$$

$$D_3[P_{s,\mathrm{d}}(\phi_{s,k})] = \frac{-2\beta m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right]}{t_{k+1} - t_k}, \tag{43}$$

$$D_4[P_{s,\mathrm{d}}(\phi_{s,k})] = \frac{2\beta m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right]}{t_{k+1} - t_k}. \tag{44}$$

由算符(6)可知

$$X_\mathrm{d}^{(1)} = \tau_k D_1 + \tau_{k+1} D_2 + \xi_{s,k} D_3 + \xi_{s,k+1} D_4. \tag{45}$$

设生成元函数 τ_k,$\xi_{s,k}$ 有如下形式:

$$\tau_k(t_k, q_{s,k}) = C_1 t_k + C_2 q_{1,k} + C_3 q_{2,k} + C_4, \tag{46}$$

$$\xi_{s,k}(t_k, q_{s,k}) = C_5 t_k + C_6 q_{1,k} + C_7 q_{2,k} + C_8. \tag{47}$$

其中 C_1—C_8 为常数.

将(33),(44)和(46),(47)式代入(45)式得到 $X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})]$,$X_\mathrm{d}^{(1)}[Q_{s,\mathrm{d}}(\phi_{s,k})]$,$X_\mathrm{d}^{(1)}[P_{s,\mathrm{d}}(\phi_{s,k})]$,并与(29)—(31)式的系数比较,有

$$C_1 = C_2 = C_3 = C_5 = C_6 = C_7 = 0, \tag{48}$$

则当

$$\tau_k(t_k, q_{s,k}) = C_4, \tag{49}$$

$$\xi_{s,k}(t_k, q_{s,k}) = C_8 \tag{50}$$

时,有

$$X_\mathrm{d}^{(1)}[L_\mathrm{d}(\phi_{s,k})] = -C_4 \gamma L_\mathrm{d}(\phi_{s,k}), \tag{51}$$

$$X_\mathrm{d}^{(1)}[Q_{s,\mathrm{d}}(\phi_{s,k})] = -C_4 \gamma Q_{s,\mathrm{d}}(\phi_{s,k}), \tag{52}$$

$$X_\mathrm{d}^{(1)}[P_{s,\mathrm{d}}(\phi_{s,k})] = -C_4 \gamma P_{s,\mathrm{d}}(\phi_{s,k}), \tag{53}$$

根据判据 1,并结合(1),(2)式,可得生成元(49),(50)是离散差分序列变质量系统(29)—(32) Mei 对称性的生成元.

取

$$\tau_k(t_k, q_{s,k}) = \xi_{s,k}(t_k, q_{s,k}) = 0, \tag{54}$$

时,$G_{Mk} = 0$,直接得到系统的平凡守恒量 $I_M = 0$.

取

$$\tau_k(t_k, q_{s,k}) = \pm 1, \tag{55}$$

$$\xi_{s,k}(t_k, q_{s,k}) = 0, \tag{56}$$

则 Mei 恒等式(18)化为

$$\begin{aligned}
&\gamma^2 m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \\
&\times \left[\left(\frac{q_{1,k+1} - q_{1,k}}{t_{k+1} - t_k}\right)^2 + \left(\frac{q_{2,k+1} - q_{2,k}}{t_{k+1} - t_k}\right)^2\right] \\
&+ \gamma(\alpha + \beta) m_0 \exp\left[-\frac{1}{2}\gamma(t_k + t_{k+1})\right] \\
&\times \left[\left(\frac{q_{1,k+1} - q_{1,k}}{t_{k+1} - t_k}\right)^2 + \left(\frac{q_{2,k+1} - q_{2,k}}{t_{k+1} - t_k}\right)^2\right] \\
&+ D_\mathrm{d}(G_{Mk}) = 0,
\end{aligned} \tag{57}$$

当 $\alpha + \beta = -\gamma$ 时,$G_{Mk} = 0$. 根据判据 2 可求出系统(29)—(32)的非平凡守恒量

$$\begin{aligned}
&\gamma m_0 \exp\left[-\frac{1}{2}\gamma(t_{k-1} + t_k)\right] \\
&\times \left[\left(\frac{q_{1,k} - q_{1,k-1}}{t_k - t_{k-1}}\right)^2 + \left(\frac{q_{2,k} - q_{2,k-1}}{t_k - t_{k-1}}\right)^2\right] \\
&+ \frac{1}{2}\gamma^2 m_0 \exp\left[-\frac{1}{2}\gamma(t_{k-1} + t_k)\right] \\
&\times \left[\frac{(q_{1,k} - q_{1,k-1})^2}{t_k - t_{k-1}} + \frac{(q_{2,k} - q_{2,k-1})^2}{t_k - t_{k-1}}\right] \\
&= \mathrm{const}.
\end{aligned}$$

而当 $\xi_{s,k}(t_k, q_{s,k}) = C_8 \neq 0$ 时,不存在离散规范函数 G_{Mk} 满足 Mei 等式(18),因此没有对应的守恒量.

4. 讨 论

本文研究了离散差分序列变质量系统的 Mei 对称性及其守恒量,主要的结果有,Mei 对称性的确定方程(16)和(17);离散 Mei 等式(18)和相应的离散守恒量(19). 这些结果可推广运用于其他带有各种特殊约束力的离散系统.

连续约束力学系统的 Mei 对称性讨论了系统 Euler-Lagrange 方程在动力学变换下形式的不变性,未涉及能量演化方程形式的不变性. 在离散系统中,用差分序列替代连续变量后,分别讨论了离散形式 Euler-Lagrange 方程和能量演化方程的形式不变性质,验证了在动力学变换下这两类方程的形式均保持不变的性质. 连续系统和离散系统通过时间步长 t_{k+1}-t_k 相联系,当时间步长 t_{k+1}-$t_k \to 0$ 时,离散系统的 Mei 对称性蜕化为连续系统的 Mei 对称性. 离散差分方程的几何意义在于建立位形空间离散点之间保动量映射、能量和辛结构的递推关系,通常也被称为动量-能量-辛积分子或保结构积分子. 连续系统 Mei 对称性的主要意义是保形式. 文中讨论的离散变质量系统两类动力学方程的数学模型为二阶变系数差分方程,离散系统的 Mei 对称性确定了离散差分方程在动力学变化下的形式不变性,增加了动量-能量-辛积分子的保形式不变的性质.

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The Mei symmetry of discrete difference sequence mechanical system with variable mass

Huang Xiao-Hong[1)] Zhang Xiao-Bo[2)] Shi Shen-Yang[2)†]

1)(*School of Physics and Electronic Information, Wenzhou University, Wenzhou* 325000, *China*)

2)(*Department of Physics, Zhejiang Sci-Tech University, Hangzhou* 310018, *China*)

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Abstract

The Mei symmetry and conserved quantity of a discrete difference sequence mechanical system with variable mass are studied in this paper. The form invariance of difference sequence equations for the discrete system under infinitesimal transformation groups is defined as Mei symmetry and the criterion when conserved quantities may be obtained from Mei symmetries is also presented. An example is given to demonstrate the applications of the results.

Keywords: discrete mechanics, system with variable mass, Mei symmetry, discrete conserved quantity

PACC: 0320, 4610

† Corresponding author. E-mail: hzshishenyang@sina.com