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双模坐标-动量积分型投影算符及其在量子光学中的应用*

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引进了幺正的双模坐标-动量积分型投影算符,利用有序算符内的积分(IWOP)技术分析了其变换特性,并导出了其正规乘积展开式.然后利用该积分型投影算符对角化了双模耦合量子谐振子体系的哈密顿量,从而求出了体系的本征能级与本征波函数.最后讨论了特例情形.

关键词:积分型投影算符,有序算符内的积分技术,坐标-动量耦合

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1. 引 言

表象与变换论在量子力学、量子光学及量子场论等许多领域有着广泛的应用.这种变换一般是通过一个幺正算符实现的.常用的幺正变换算符有平移算符、压缩算符、转动算符等.另一方面,压缩态已成为量子光学领域中一个极为重要的热点.文献[1—11]对表象变换及其应用作了广泛的探讨.文献[12]引进并研究了幺正的单模积分型压缩算符,文献[13]引进并研究了幺正的四模积分型压缩算符,文献[14]引进并研究了幺正的 N 模压缩算符.本文将基于经典辛变换 $(q_1, p_2) \to (A_{11} q_1 + A_{12} p_2, A_{21} q_1 + A_{22} p_2)$ 引进一个幺正的双模坐标-动量积分型投影算符,并利用有序算符内的积分(IWOP)技术[12]分析其变换特性,导出了该双模坐标-动量积分型投影算符的正规乘积展开式,并计算了压缩真空态的量子涨落;然后利用该坐标-动量积分型投影算符对角化了双模耦合量子谐振子体系的哈密顿量,从而求出了体系的本征能级与本征波函数.

2. 双模积分型投影算符

双模坐标-动量积分型投影算符定义为

$$\hat{U} = \iint_{-\infty}^{\infty} \mathrm{d}q_1 \mathrm{d}p_2 \left| \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} q_1 \\ p_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ p_2 \end{pmatrix} \right|, \tag{1}$$

其中,A_{11}、A_{12}、A_{21} 及 A_{22} 为实数,$\left|\begin{pmatrix} q_1 \\ p_2 \end{pmatrix}\right\rangle$ 为双模坐标算符 $\hat{q}_1$ 与动量算符 $\hat{p}_2$ 的共同本征态,在双模 Fock 空间中的展开式为

$$\begin{aligned}\left|\begin{pmatrix} q_1 \\ p_2 \end{pmatrix}\right\rangle &\equiv |q_1, p_2\rangle \\ &= \pi^{-1/2} \exp\left(-\frac{1}{2} q_1^2 - \frac{1}{2} p_2^2 + \sqrt{2} q_1 \hat{a}_1^\dagger \right. \\ &\quad \left. + \mathrm{i}\sqrt{2} p_2 \hat{a}_2^\dagger - \frac{\hat{a}_1^{\dagger 2} - \hat{a}_2^{\dagger 2}}{2} \right) |00\rangle,\end{aligned} \tag{2}$$

$$\begin{aligned}&\left|\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} q_1 \\ p_2 \end{pmatrix}\right\rangle \\ &= \pi^{-1/2} \exp\left\{ -\frac{(A_{11} q_1 + A_{12} p_2)^2}{2} \right. \\ &\quad - \frac{(A_{21} q_1 + A_{22} p_2)^2}{2} + \sqrt{2}(A_{11} q_1 + A_{12} p_2) \hat{a}_1^\dagger \\ &\quad \left. + \mathrm{i}\sqrt{2}(A_{21} q_1 + A_{22} p_2) \hat{a}_2^\dagger - \frac{\hat{a}_1^{\dagger 2} - \hat{a}_2^{\dagger 2}}{2} \right\} |00\rangle,\end{aligned} \tag{3}$$

这里 $|00\rangle$ 为双模真空态,$\hat{a}_i^\dagger$、$\hat{a}_i$ 分别为产生算符、湮灭算符,它们与坐标算符 $\hat{q}_i$、动量算符 $\hat{p}_i$ 的关系为 $\hat{q}_i = (\hat{a}_i + \hat{a}_i^\dagger)/\sqrt{2}$,$\hat{p}_i = (\hat{a}_i - \hat{a}_i^\dagger)/\mathrm{i}\sqrt{2}$ $(i = 1, 2)$.

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如果

$$A_{11}A_{22}-A_{12}A_{21}=1, \tag{4}$$

那么 $\hat{U}$ 的幺正性可利用 IWOP 技术 $:\frac{\partial}{\partial\hat{a}_i^\dagger}f(\hat{a}_i^\dagger,\hat{a}_i):=[\hat{a},:f(\hat{a}_i^\dagger,\hat{a}_i):]$、$\hat{a}_i|00\rangle=0$ 以及双模真空态投影算符的正规乘积形式 $|00\rangle\langle00|=:\exp\{-\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2\}:$[12] 来证明，这里 $:$ 表示正规乘积，即

$$\begin{aligned}\hat{U}\hat{U}^\dagger &= \iiiint_{-\infty}^{\infty}\mathrm{d}q_1'\mathrm{d}p_2'\mathrm{d}q_1\mathrm{d}p_2 \\ &\left|\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}q_1'\\p_2'\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q_1'\\p_2'\end{pmatrix}\right| \\ &\left|\begin{pmatrix}q_1\\p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}q_1\\p_2\end{pmatrix}\right| \\ &=\iint_{-\infty}^{\infty}\mathrm{d}q_1\mathrm{d}p_2\left|\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}q_1\\p_2\end{pmatrix}\right\rangle \\ &\left\langle\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}q_1\\p_2\end{pmatrix}\right| \\ &=\pi^{-1}:\iint_{-\infty}^{\infty}\mathrm{d}q_1\mathrm{d}p_2\exp\{-(A_{11}q_1+A_{12}p_2)^2 \\ &-(A_{21}q_1+A_{22}p_2)^2 \\ &+\sqrt{2}(A_{11}q_1+A_{12}p_2)(\hat{a}_1^\dagger+\hat{a}_1) \\ &+\mathrm{i}\sqrt{2}(A_{21}q_1+A_{22}p_2)(\hat{a}_2^\dagger-\hat{a}_2) \\ &-(\hat{a}_1^{\dagger2}+\hat{a}_1^2-\hat{a}_2^{\dagger2}-\hat{a}_2^2)/2-\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2\}: \\ &=:\exp\{0\}:=1=\hat{U}^\dagger\hat{U},\end{aligned} \tag{5}$$

这里已经使用了积分公式 $\int_{-\infty}^{\infty}\exp\{-\alpha x^2+\beta x\}\mathrm{d}x=\sqrt{\pi/\alpha}\exp\{\beta^2/4\alpha\}$，$\mathrm{Re}(\alpha)>0$.

现在分析 $\hat{U}$ 对坐标算符 $\hat{q}_1$ 的变换特性，即

$$\begin{aligned}\hat{U}\hat{q}_1\hat{U}^\dagger &= \iiiint_{-\infty}^{\infty}\mathrm{d}q_1'\mathrm{d}p_2'\mathrm{d}q_1\mathrm{d}p_2 \\ &\left|\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}q_1'\\p_2'\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q_1'\\p_2'\end{pmatrix}\right| \\ &\hat{q}_1\left|\begin{pmatrix}q_1\\p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}q_1\\p_2\end{pmatrix}\right| \\ &=\iint_{-\infty}^{\infty}q_1\mathrm{d}q_1\mathrm{d}p_2\left|\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}q_1\\p_2\end{pmatrix}\right\rangle \\ &\left\langle\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}q_1\\p_2\end{pmatrix}\right| \\ &=\pi^{-1}:\iint_{-\infty}^{\infty}q_1\mathrm{d}q_1\mathrm{d}p_2 \\ &\times\exp\{-(A_{11}q_1+A_{12}p_2)^2 \\ &-(A_{21}q_1+A_{22}p_2)^2 \\ &+\sqrt{2}(A_{11}q_1+A_{12}p_2)(\hat{a}_1^\dagger+\hat{a}_1) \\ &+\mathrm{i}\sqrt{2}(A_{21}q_1+A_{22}p_2)(\hat{a}_2^\dagger-\hat{a}_2) \\ &-\frac{\hat{a}_1^{\dagger2}+\hat{a}_1^2-\hat{a}_2^{\dagger2}-\hat{a}_2^2}{2}-\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2\}: \\ &=A_{22}(\hat{a}_1+\hat{a}_1^\dagger)/\sqrt{2}-A_{12}(\hat{a}_2-\hat{a}_2^\dagger)/\mathrm{i}\sqrt{2} \\ &=A_{22}\hat{q}_1-A_{12}\hat{p}_2,\end{aligned} \tag{6}$$

这里已经使用了积分公式

$$\int_{-\infty}^{\infty}x\exp\{-\alpha x^2+\beta x\}\mathrm{d}x=\frac{\beta}{2\alpha}\sqrt{\pi/\alpha}\exp\{\beta^2/4\alpha\},(\mathrm{Re}(\alpha)>0).$$

同理可得

$$\begin{aligned}\hat{U}\hat{q}_2\hat{U}^\dagger&=A_{22}\hat{q}_2-A_{12}\hat{p}_1,\\ \hat{U}\hat{p}_1\hat{U}^\dagger&=A_{11}\hat{p}_1-A_{21}\hat{q}_2,\\ \hat{U}\hat{p}_2\hat{U}^\dagger&=A_{11}\hat{p}_2-A_{21}\hat{q}_1,\\ \hat{U}^\dagger\hat{q}_1\hat{U}&=A_{11}\hat{q}_1+A_{12}\hat{p}_2,\\ \hat{U}^\dagger\hat{q}_2\hat{U}&=A_{11}\hat{q}_2+A_{12}\hat{p}_1,\\ \hat{U}^\dagger\hat{p}_1\hat{U}&=A_{22}\hat{p}_1+A_{21}\hat{q}_2,\\ \hat{U}^\dagger\hat{p}_2\hat{U}&=A_{22}\hat{p}_2+A_{21}\hat{q}_1.\end{aligned} \tag{7}$$

由(6)，(7)式可见，通过 $\hat{U}$ 实现的幺正变换一般来说既包含压缩变换，又包含转动变换.

3. $\hat{U}$ 的正规乘积展开式

利用 IWOP 技术，完成(1)式的积分，可以得到投影算符 $\hat{U}$ 的正规乘积展开式

$$\begin{aligned}\hat{U}&=\frac{2}{\sqrt{L}}\exp\left\{\frac{1}{2L}[(A_{11}^2+A_{12}^2-A_{21}^2-A_{22}^2)(\hat{a}_1^{\dagger2}+\hat{a}_2^{\dagger2})\right.\\ &\left.+4\mathrm{i}(A_{11}A_{21}+A_{12}A_{22})\hat{a}_1^\dagger\hat{a}_2^\dagger]\right\}\\ &\times:\exp\left\{\frac{2(A_{11}+A_{22})-L}{L}(\hat{a}_1^\dagger\hat{a}_1+\hat{a}_2^\dagger\hat{a}_2)\right.\\ &\left.+\mathrm{i}\frac{2(A_{21}-A_{12})}{L}(\hat{a}_1^\dagger\hat{a}_2+\hat{a}_2^\dagger\hat{a}_1)\right\}:\\ &\times\exp\left\{\frac{1}{2L}[(A_{12}^2+A_{22}^2-A_{11}^2-A_{21}^2)(\hat{a}_1^2+\hat{a}_2^2)\right.\\ &\left.+4\mathrm{i}(A_{11}A_{12}+A_{21}A_{22})\hat{a}_1\hat{a}_2\right\},\end{aligned}$$

这里，$L=2+A_{11}^2+A_{12}^2+A_{21}^2+A_{22}^2$. 把 $\hat{U}$ 的正规乘积形式作用于真空态 $|00\rangle$，得到压缩真空态

$$
\begin{aligned}
|00\rangle_U &= \hat{U}|00\rangle \\
&= \frac{2}{\sqrt{L}}\exp\left\{\frac{1}{2L}\left[(A_{11}^2 + A_{12}^2 - A_{21}^2 - A_{22}^2)(\hat{a}_1^{\dagger 2} + \hat{a}_2^{\dagger 2})\right.\right. \\
&\quad \left.\left. + 4\mathrm{i}(A_{11}A_{21} + A_{12}A_{22})\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}\right]\right\}|00\rangle,
\end{aligned} \tag{8}
$$

为描述这个态的不确定性,我们计算量子涨落.利用(7)式,可以得到

$$
\begin{aligned}
{}_U\langle 00|\hat{q}_1|00\rangle_U &= 0, \\
{}_U\langle 00|\hat{q}_2|00\rangle_U &= 0, \\
{}_U\langle 00|\hat{p}_1|00\rangle_U &= 0, \\
{}_U\langle 00|\hat{p}_2|00\rangle_U &= 0, \\
{}_U\langle 00|\hat{q}_1^2|00\rangle_U &= (A_{11}^2 + A_{12}^2)/2, \\
{}_U\langle 00|\hat{q}_2^2|00\rangle_U &= (A_{11}^2 + A_{12}^2)/2, \\
{}_U\langle 00|\hat{p}_1^2|00\rangle_U &= (A_{21}^2 + A_{22}^2)/2, \\
{}_U\langle 00|\hat{p}_2^2|00\rangle_U &= (A_{21}^2 + A_{22}^2)/2,
\end{aligned}
$$

$$
\begin{aligned}
\Delta q_1 \cdot \Delta p_1 &= \Delta q_2 \cdot \Delta p_2 \\
&= \sqrt{(A_{11}^2 + A_{12}^2)(A_{21}^2 + A_{22}^2)}/2 \\
&= \sqrt{1 + (A_{11}A_{21} + A_{21}A_{22})^2}/2 \geqslant 1/2.
\end{aligned} \tag{9}
$$

(9)式表明,当 $A_{11}A_{21} + A_{21}A_{22} = 0$ 时,$|00\rangle_U$ 态的不确定性最小.

4. 应用举例

谐振子,尤其是耦合谐振子,在量子体系中占有着极其重要的地位.但是,耦合谐振子的动力学问题很难在坐标表象中直接严格求解.作为双模坐标-动量积分型投影算符的一个应用,我们研究双模耦合量子谐振子体系的动力学.一类双模坐标-动量耦合量子谐振子体系的哈密顿量为

$$
\begin{aligned}
\hat{H} &= \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + \frac{1}{2}m_1\omega_1^2\hat{q}_1^2 \\
&\quad + \frac{1}{2}m_2\omega_2^2\hat{q}_2^2 - \lambda_1\hat{q}_2\hat{p}_1 - \lambda_2\hat{q}_1\hat{p}_2,
\end{aligned} \tag{10}
$$

从量子光学的角度来说,该哈密顿量包含着描述参量转换与拉曼转换过程的耦合项.为不失一般性,假设 $\omega_1 \geqslant \omega_2$,$\lambda_1 \geqslant \lambda_2 > 0$.利用幺正算符 $\hat{U}$ 对 $\hat{H}$ 作量子幺正变换并结合(6)和(7)式,于是得到

$$
\begin{aligned}
U\hat{H}\hat{U}^{\dagger} &= \left(\frac{A_{11}^2}{2m_1} + \frac{1}{2}m_2\omega_2^2A_{12}^2 + \lambda_1A_{11}A_{12}\right)\hat{p}_1^2 \\
&\quad + \left(\frac{A_{11}^2}{2m_2} + \frac{1}{2}m_1\omega_1^2A_{12}^2 + \lambda_2A_{11}A_{12}\right)\hat{p}_2^2 \\
&\quad + \left(\frac{A_{21}^2}{2m_2} + \frac{1}{2}m_1\omega_1^2A_{22}^2 + \lambda_2A_{21}A_{22}\right)\hat{q}_1^2 \\
&\quad + \left(\frac{A_{21}^2}{2m_1} + \frac{1}{2}m_2\omega_2^2A_{22}^2 + \lambda_1A_{21}A_{22}\right)\hat{q}_2^2 \\
&\quad - \left(\frac{A_{11}A_{21}}{m_1} + m_2\omega_2^2A_{12}A_{22}\right. \\
&\quad \left. - \lambda_1A_{11}A_{22} - \lambda_1A_{12}A_{21}\right)\hat{q}_2\hat{p}_1 \\
&\quad - \left(\frac{A_{11}A_{21}}{m_2} + m_1\omega_1^2A_{12}A_{22}\right. \\
&\quad \left. - \lambda_2A_{11}A_{22} - \lambda_2A_{12}A_{21}\right)\hat{q}_1\hat{p}_2.
\end{aligned} \tag{11}
$$

令

$$
\begin{aligned}
&\frac{A_{11}A_{21}}{m_1} + m_2\omega_2^2A_{12}A_{22} \\
&\quad - \lambda_1A_{11}A_{22} - \lambda_1A_{12}A_{21} = 0, \\
&\frac{A_{11}A_{21}}{m_2} + m_1\omega_1^2A_{12}A_{22} \\
&\quad - \lambda_2A_{11}A_{22} - \lambda_2A_{12}A_{21} = 0,
\end{aligned} \tag{12}
$$

若取 $A_{11} = 1$,则由(4),(12)式可解得

$$
\begin{aligned}
A_{12} &= \frac{\omega_1^2 - \omega_2^2 - \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4(\lambda_1m_1 - \lambda_2m_2)(\lambda_1m_1\omega_1^2 - \lambda_2m_2\omega_2^2)/m_1m_2}}{2(\lambda_1m_1\omega_1^2 - \lambda_2m_2\omega_2^2)}, \\
A_{21} &= \frac{\lambda_1m_1\omega_1^2 - \lambda_2m_2\omega_2^2}{\sqrt{(\omega_1^2 - \omega_2^2)^2 + 4(\lambda_1m_1 - \lambda_2m_2)(\lambda_1m_1\omega_1^2 - \lambda_2m_2\omega_2^2)/m_1m_2}}, \\
A_{22} &= \frac{1}{2} + \frac{\omega_1^2 - \omega_2^2}{2\sqrt{(\omega_1^2 - \omega_2^2)^2 + 4(\lambda_1m_1 - \lambda_2m_2)(\lambda_1m_1\omega_1^2 - \lambda_2m_2\omega_2^2)/m_1m_2}},
\end{aligned} \tag{13}
$$

代入(11)式,于是有

$$\hat{U}\hat{H}\hat{U}^{\dagger}=\frac{1}{2\mu_1}\hat{p}_1^2+\frac{1}{2\mu_2}\hat{p}_2^2+\frac{1}{2}\mu_1\Omega_1^2\hat{q}_1^2+\frac{1}{2}\mu_2\Omega_2^2\hat{q}_2^2,\tag{14}$$

其中,

$$\mu_1=m_1/\{A_{11}^2+m_1m_2\omega_2^2A_{12}^2+2\lambda_1m_1A_{11}A_{12}\},$$

$$\mu_2=m_2/\{A_{11}^2+m_1m_2\omega_1^2A_{12}^2+2\lambda_2m_2A_{11}A_{12}\},$$

$$\Omega_1=\sqrt{(A_{21}^2+m_1m_2\omega_1^2A_{22}^2+2\lambda_2m_2A_{21}A_{22})/m_2\mu_1},$$

$$\Omega_2=\sqrt{(A_{21}^2+m_1m_2\omega_2^2A_{22}^2+2\lambda_1m_1A_{21}A_{22})/m_1\mu_2}.$$

至此,已实现了哈密顿量的对角化.设 $\hat{H}$ 的本征矢为 $|E\rangle$,用双模坐标本征态 $\langle q_1q_2|$ 和 $\hat{U}|E\rangle$ 夹乘(14)式,得

$$\begin{aligned}&E\langle q_1q_2|\hat{U}|E\rangle\\&=-\frac{\hbar^2}{2\mu_1}\frac{\partial^2}{\partial q_1^2}\langle q_1q_2|\hat{U}|E\rangle\\&\quad+\frac{1}{2}\mu_1\Omega_1^2q_1^2\langle q_1q_2|\hat{U}|E\rangle\\&\quad-\frac{\hbar^2}{2\mu_2}\frac{\partial^2}{\partial q_2^2}\langle q_1q_2|\hat{U}|E\rangle\\&\quad+\frac{1}{2}\mu_2\Omega_2^2q_2^2\langle q_1q_2|\hat{U}|E\rangle,\end{aligned}\tag{15}$$

这里已恢复了普朗克常数 $\hbar$.(15)式就是定态薛定谔方程,$\langle q_1q_2|\hat{U}|E\rangle$ 为波函数,其解为

$$\begin{aligned}&E_{n_1n_2}=\sum_{i=1}^{2}\left\{\left(n_i+\frac{1}{2}\right)\hbar\Omega_i\right\},\\&\langle q_1q_2|\hat{U}|E_{n_1n_2}\rangle\\&\quad=\prod_{i=1}^{2}\left\{N_{n_i}\exp\left(-\frac{\alpha_i^2}{2}q_i^2\right)H_{n_i}(\alpha_iq_i)\right\},\end{aligned}\tag{16}$$

其中,$n_i=0,1,2,\cdots$;$\alpha_i=\sqrt{\mu_i\Omega_i/\hbar}$;$N_{n_i}=\sqrt{\alpha_i/(\sqrt{\pi}2^{n_i}\cdot n_i!)}$ 为归一化常数;$H_{n_i}(\alpha_iq_i)$ 为厄米多项式.

为求出波函数 $\langle q_1q_2|E_{n_1n_2}\rangle$,先计算 $\hat{U}|q_1q_2\rangle$,即

$$\begin{aligned}\hat{U}|q_1q_2\rangle&=\iint_{-\infty}^{\infty}\mathrm{d}q_1'\mathrm{d}q_2'\left|\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}q_1'\\p_2'\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q_1'\\p_2'\end{pmatrix}\middle|\begin{pmatrix}q_1\\q_2\end{pmatrix}\right\rangle\\&=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{d}p_2'\exp(\mathrm{i}p_2'q_2/\hbar)\left|\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}q_1\\p_2\end{pmatrix}\right\rangle,\end{aligned}\tag{17}$$

于是得到

$$\begin{aligned}\langle q_1q_2|E_{n_1n_2}\rangle&=\langle q_1q_2|\hat{U}^{\dagger}\hat{U}|E_{n_1n_2}\rangle\\&=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{d}p_2'\exp(\mathrm{i}p_2'q_2/\hbar)\left\langle\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}q_1\\p_2'\end{pmatrix}\middle|\hat{U}|E_{n_1n_2}\right\rangle\\&=\frac{N_{n_1}N_{n_2}}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\exp(\mathrm{i}p_2'q_2/\hbar)\\&\quad\times\exp\left[-\frac{\alpha_1^2}{2}(A_{11}q_1+A_{12}p_2')^2\right]\\&\quad\times H_{n_1}[\alpha_1(A_{11}q_1+A_{12}p_2')]\\&\quad\times\exp\left[-\frac{\alpha_2^2}{2}(A_{21}q_1+A_{22}p_2')^2\right]\\&\quad\times H_{n_2}[\alpha_2(A_{21}p_1+A_{22}p_2')]\mathrm{d}p_2',\end{aligned}\tag{18}$$

式中,A_{11},A_{12},A_{21},DA_{22} 的数值由(13)式给出.

特别地,当 $\lambda_1=\lambda_2=0$ 时,由(13)式得 $A_{11}=A_{22}=1$,$A_{12}=A_{21}=0$,进一步可求得 $\mu_1=m_1$,$\mu_2=m_2$,$\Omega_1=\omega_1$,$\Omega_2=\omega_2$,这正是无耦合的情形;当 $m_1=m_2\equiv m$,$\omega_1=\omega_2\equiv\omega$,$\lambda_1$,$\lambda_2\neq0$ 时,由(13)式得 $A_{11}=1$,$A_{12}=-1/m\omega$,$A_{21}=m\omega/2$,$A_{22}=1/2$,进一步可求得 $\mu_1=m\omega/2(\omega-\lambda_1)$,$\mu_2=m\omega/2(\omega-\lambda_2)$,$\Omega_1=\omega\sqrt{(1-\lambda_1/\omega)(1+\lambda_2/\omega)}$,$\Omega_2=\omega\sqrt{(1+\lambda_1/\omega)(1-\lambda_2/\omega)}$,这是质量与角频率均相同、具有运动耦合的两个量子谐振子体系的情形.

5. 结 论

我们引进了一个双模坐标-动量积分型投影算符,证明了它的幺正性,分析了它的变换特性,提供了一种由经典辛变换向量子幺正变换过渡的捷径,为严格求解双模坐标-动量耦合体系的动力学问题提供了一种行之有效的方法.在本文的证明与分析中,IWOP 技术起着极为重要的作用.

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Two-mode integral form projection operator and its application in quantum optics*

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Abstract

A two-mode integral form projection unitary operator is introduced. Using the technique of integration within an ordered product (IWOP) of operators, the transformation property of the integral form projection operator is studied, and its normal ordered expression is derived. Then the Hamiltonian of two-mode quantum harmonic oscillator system with coordinate-momentum coupling is diagonalized by virtue of the integral form projection operator, there by the eigenenergy and eigenfunction of the system are exactly solved. Lastly, a special case is discussed.

Keywords: integral form projection operator, integration within an ordered product technique, coordinate-momentum coupling
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