

# 基于双模相干-纠缠态表象的算符恒等式构造法\*

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利用最近引进的相干-纠缠态  $|x, \alpha\rangle$  表象和有序算符内的积分技术, 给出了关于双模光场正交分量  $(X_1 + X_2)/\sqrt{2}$  和参数化的正交分量  $(\mu X_1 + \nu X_2)/\lambda$  的函数的若干正规乘积展开式, 这对于研究场的高阶压缩行为和构建广义压缩态是有用的.

关键词: 双模相干-纠缠态, 算符恒等式, 有序算符内的积分技术

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## 1. 引 言

范洪义等首次提出了相干-纠缠态的概念<sup>[1]</sup>, 即此态既具有相干态的性质, 也具有纠缠态的行为. 此态矢量在 Fock 空间的表达式为

$$|x, \alpha\rangle = \exp\left[-\frac{x^2}{2} - \frac{1}{4}|\alpha|^2 + \left(x + \frac{\alpha}{2}\right)a_1^\dagger + \left(x - \frac{\alpha}{2}\right)a_2^\dagger - \frac{1}{4}(a_1^\dagger + a_2^\dagger)^2\right] |00\rangle \quad (1)$$

其中  $\alpha$  是复数  $\alpha = \alpha_1 + i\alpha_2$ ,  $x$  为实数,  $a_i^\dagger$  和  $a_i$  ( $i = 1, 2$ ) 分别代表光子的产生算符与消灭算符, 它们之间的对易关系是  $[a_i, a_j^\dagger] = \delta_{ij}$ . 利用此对易关系和公式

$$[a_i, f(a_1^\dagger, a_2^\dagger)] = \frac{\partial}{\partial a_i} f(a_1^\dagger, a_2^\dagger), \quad (2)$$

可以得到

$$a_1 |x, \alpha\rangle = \left[x + \frac{\alpha}{2} - \frac{1}{2}(a_1^\dagger + a_2^\dagger)\right] |x, \alpha\rangle \quad (3)$$

$$a_2 |x, \alpha\rangle = \left[x - \frac{\alpha}{2} - \frac{1}{2}(a_1^\dagger + a_2^\dagger)\right] |x, \alpha\rangle \quad (4)$$

结合(3)和(4)式, 可得相干-纠缠态  $|x, \alpha\rangle$  满足以下本征方程, 即

$$(a_1 - a_2) |x, \alpha\rangle = \alpha |x, \alpha\rangle,$$

$$\frac{1}{2}(X_1 + X_2) |x, \alpha\rangle = \frac{1}{\sqrt{2}}x |x, \alpha\rangle, \quad (5)$$

及对易关系

$$[a_1 - a_2, X_1 + X_2] = 0, \quad (6)$$

其中  $X_i = \frac{1}{\sqrt{2}}(a_i + a_i^\dagger)$  ( $i = 1, 2$ ) 是光场的正交分量之一. 由方程(5)可见  $|x, \alpha\rangle$  具有相干态的性质, 而由(1)可知它具有纠缠的行为. 为了更明显地看出这两点, 我们写出  $|x, \alpha\rangle$  的部分正交性与部分非正交性

$$x' \langle \alpha' | x, \alpha\rangle = \sqrt{\pi} \exp\left[-\frac{1}{4}(|\alpha|^2 + |\alpha'|^2) + \frac{1}{2}\alpha\alpha'^*\right] \delta(x' - x). \quad (7)$$

用光分束器及两束不同模的激光以及单模压缩态可以产生此相干-纠缠态<sup>[1]</sup>. 本文将用它的完备性关系

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d^2\alpha}{2\pi} |x, \alpha\rangle \langle x, \alpha| = \int_{-\infty}^{\infty} \frac{d^2\alpha}{2\pi} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} : \exp\left[-x^2 - \frac{1}{2}|\alpha|^2 + \left(x + \frac{\alpha}{2}\right)a_1^\dagger + \left(x - \frac{\alpha}{2}\right)a_2^\dagger + \left(x + \frac{\alpha^*}{2}\right)a_1 + \left(x - \frac{\alpha^*}{2}\right)a_2\right] \exp\left[-\frac{1}{4}(a_1^\dagger + a_2^\dagger)^2 - \frac{1}{4}(a_1 + a_2)^2\right]$$

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$$- a_1^\dagger a_1 - a_2^\dagger a_2] := 1 \quad (8)$$

(上式中,  $\dots$  表示算符的正规排序, 在正规排序内所有的算符被看成是对易的, 且在积分过程中可当作参数来对待<sup>[2-4]</sup>) 来导出关于双模光场正交分量  $(X_1 + X_2)/\sqrt{2}$  的函数的若干正规乘积展开式, 这对于研究场的高阶压缩<sup>[5,6]</sup> 行为和构建广义压缩态是有用的.

## 2. $(X_1 + X_2)^n$ 的正规乘积展开

由(5)式和完备性(8)式及有序算符内的积分技术<sup>[2-4]</sup>, 有

$$\left[ \frac{1}{\sqrt{2}}(X_1 + X_2) \right]^n = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d^2\alpha}{2\pi} x^n |x, \alpha\rangle \langle x, \alpha|, \quad (9)$$

用双模真空投影算符的正规乘积式  $|00\rangle\langle 00| = \exp[-a_1^\dagger a_1 - a_2^\dagger a_2]$  和相干纠缠态  $|x, \alpha\rangle$  的表达式(1)对  $\int_{-\infty}^{\infty} d^2\alpha$  积分, 积分公式为<sup>[7]</sup>

$$\int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi} \exp[\lambda|\alpha|^2 + \mu\alpha + \nu\alpha^*] = -\frac{1}{\lambda} e^{-\mu\nu/\lambda}, \quad \text{Re}\lambda < 0, \quad (10)$$

立即得到

$$\left[ \frac{X_1 + X_2}{\sqrt{2}} \right]^n = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} x^n \exp\left[-\left(x - \frac{a_1^\dagger + a_1 + a_2^\dagger + a_2}{2}\right)^2\right] := \quad (11)$$

再利用积分公式

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} x^n \exp[-\sigma(x - \lambda)^2] = \frac{1}{\sqrt{\sigma^{n+1}}} \sum_{k=0}^{n/2} \frac{n!}{2^{2k} k! (n-2k)!} (\sigma^{1/2} \lambda)^{n-2k}, \quad \text{Re}\sigma > 0, \quad (12)$$

并注意到  $X_i$  与光子产生算符与湮灭算符的关系, 可进一步将(11)式改写成:

$$\left[ \frac{X_1 + X_2}{\sqrt{2}} \right]^n = \sum_{k=0}^{n/2} \frac{n!}{2^{2k} k! (n-2k)!}$$

$$\begin{aligned} & \times \left( \frac{a_1^\dagger + a_1 + a_2^\dagger + a_2}{2} \right)^{n-2k} := \\ & = \sum_{k=0}^{n/2} \frac{n!}{2^{2k} k! (n-2k)!} \left[ \frac{X_1 + X_2}{\sqrt{2}} \right]^{n-2k} := \\ & = \frac{1}{(2i)^n} \sum_{k=0}^{n/2} \frac{(-)^k n!}{k! (n-2k)!} \left[ 2i \frac{X_1 + X_2}{\sqrt{2}} \right]^{n-2k} := \\ & = \frac{1}{(2i)^n} :H_n\left(i \frac{X_1 + X_2}{\sqrt{2}}\right):, \end{aligned} \quad (13)$$

其中  $H_n(x)$  是单变量厄米多项式, 定义为

$$H_n(x) = \sum_{k=0}^{n/2} \frac{(-)^k n!}{k! (n-2k)!} (2x)^{n-2k}. \quad (14)$$

(13)式即是其正规乘积展开式, 也是一个新的算符恒等式. 由相干态的性质<sup>[8-10]</sup>

$$\begin{aligned} a_1 |z_1\rangle &= z_1 |z_1\rangle, \\ z_1 |z_1\rangle &= 1, \end{aligned}$$

$$z_1 |X_1|z_1\rangle = \frac{z_1 + z_1^*}{\sqrt{2}} = \sqrt{2} \text{Re}z_1. \quad (15)$$

我们可以立即求出它的相干态期望值

$$\begin{aligned} & z_1 |z_2\rangle \left[ \frac{X_1 + X_2}{\sqrt{2}} \right]^n |z_1, z_2\rangle \\ & = \frac{1}{(2i)^n} z_1 |z_2\rangle :H_n\left(i \frac{X_1 + X_2}{\sqrt{2}}\right): |z_1, z_2\rangle \\ & = \frac{1}{(2i)^n} H_n[\sqrt{2}(\text{Re}z_1 + \text{Re}z_2)]. \end{aligned} \quad (16)$$

## 3. $\exp\left[\frac{i\gamma}{2}(X_1 + X_2)^2\right]$ 的正规乘积展开

在此我们进一步求么正算符  $\exp\left[\frac{i\gamma}{2}(X_1 + X_2)^2\right]$  的正规乘积展开. 用(5)式和完备性(8)式又可得

$$\begin{aligned} & \exp\left[\frac{i\gamma}{2}(X_1 + X_2)^2\right] \\ & = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d^2\alpha}{2\pi} \exp(i\gamma x^2) |x, \alpha\rangle \langle x, \alpha|. \end{aligned} \quad (17)$$

用有序算符内的积分技术进行积分后得到它的正规乘积式

$$\begin{aligned} & \exp\left[\frac{i\gamma}{2}(X_1 + X_2)^2\right] \\ & = \frac{1}{\sqrt{1-i\gamma}} \exp\left[\frac{i\gamma}{4(1-i\gamma)}(a_1^\dagger + a_2^\dagger)^2\right] \\ & \quad \times \exp\left[\frac{i\gamma}{2(1-i\gamma)}(a_1^\dagger a_1 + a_2^\dagger a_2 + a_1^\dagger a_2 + a_2^\dagger a_1) - a_1^\dagger a_1 - a_2^\dagger a_2\right] := \end{aligned}$$

$$\times \exp\left[\frac{i\gamma}{4(1-i\gamma)}(a_1 + a_2)^2\right]. \quad (18)$$

将(18)式作用于双模真空态得到

$$\begin{aligned} & \exp\left[\frac{i\gamma}{2}(X_1 + X_2)^2\right] |00\rangle \\ &= \frac{1}{\sqrt{1-i\gamma}} \exp\left[\frac{i\gamma(a_1^\dagger + a_2^\dagger)^2}{4(1-i\gamma)}\right] |00\rangle, \quad (19) \end{aligned}$$

可见它是一个单-双模组合压缩态.

#### 4. $H_n[(X_1 + X_2)/\sqrt{2}]$ 的正规乘积展开

另一方面,我们又可以计算厄米多项式算符  $H_n[(X_1 + X_2)/\sqrt{2}]$  的正规乘积展开.利用式(5)和完备性(8)及以下积分公式<sup>[11]</sup>

$$\int_{-\infty}^{\infty} e^{-(x-y)^2} H_n(x) dx = \sqrt{\pi} (2y)^n, \quad (20)$$

可得

$$\begin{aligned} & H_n\left[\frac{1}{\sqrt{2}}(X_1 + X_2)\right] \\ &= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d^2\alpha}{2\pi} H_n(x) |x, \alpha\rangle \langle x, \alpha| \\ &= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} :H_n(x) \\ & \quad \times \exp\left[-\left(x - \frac{a_1^\dagger + a_1 + a_2^\dagger + a_2}{2}\right)\right] : \\ &= (a_1^\dagger + a_1 + a_2^\dagger + a_2)^n : \\ &= 2^{n/2} (X_2 + X_2)^n :. \quad (21) \end{aligned}$$

由(21)式我们可进一步导出

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\sqrt{2}t)^n}{n!} H_n\left(\frac{X_1 + X_2}{\sqrt{2}}\right) \\ &= \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} (X_1 + X_2)^n : \\ &= \exp[2t(X_1 + X_2)]:. \quad (22) \end{aligned}$$

(22)式是关于厄米多项式的又一个算符恒等式.

#### 5. 参数化的算符恒等式

胡利云等将  $|x, \alpha\rangle$  推广得到带参数的双模相干-纠缠态  $|x, \alpha, \mu, \nu\rangle$ <sup>[12,13]</sup> 其在 Fock 空间的表达式为

$$\begin{aligned} |x, \alpha, \mu, \nu\rangle &\equiv \exp\left[-\frac{x^2}{2} - \frac{1}{4}|\alpha|^2 + \frac{\sqrt{2}}{\lambda}(x\mu + \frac{\alpha\nu}{2})a_1^\dagger \right. \\ & \quad \left. + \frac{\sqrt{2}}{\lambda}(x\nu - \frac{\alpha\mu}{2})a_2^\dagger \right] \end{aligned}$$

$$- \frac{1}{2\lambda^2}(\mu a_1^\dagger + \nu a_2^\dagger)^2] |00\rangle, \quad (23)$$

其中  $\mu, \nu$  是两个独立的参数,  $\lambda = \sqrt{\mu^2 + \nu^2}$ ,  $x$  是实数,  $\alpha = \alpha_1 + i\alpha_2$  是复数.它具有完备性

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d^2\alpha}{2\pi} |x, \alpha, \mu, \nu\rangle \langle x, \alpha| = 1, \quad (24)$$

和部分正交性和部分非正交性

$$\begin{aligned} & \langle x', \alpha', \mu', \nu' | x, \alpha, \mu, \nu\rangle \\ &= \sqrt{\pi} \exp\left[-\frac{1}{4}(|\alpha|^2 + |\alpha'|^2) + \frac{1}{2}\alpha\alpha'^*\right] \delta(x' - x). \quad (25) \end{aligned}$$

用光子产生算符  $a_i^\dagger$  与消灭算符  $a_i$  的对易式  $[a_i, a_j^\dagger] = \delta_{ij}$ , 我们可以得到本征方程

$$\begin{aligned} (\mu X_1 + \nu X_2) |x, \alpha, \mu, \nu\rangle &= \lambda x |x, \alpha, \mu, \nu\rangle, \\ (\nu a_1 - \mu a_2) |x, \alpha, \mu, \nu\rangle &= \frac{\lambda\alpha}{\sqrt{2}} |x, \alpha, \mu, \nu\rangle. \quad (26) \end{aligned}$$

由此可见,  $|x, \alpha, \mu, \nu\rangle$  是  $(\mu X_1 + \nu X_2)$  和  $(\nu a_1 - \mu a_2)$  的共同本征矢,且有对易关系

$$[\mu X_1 + \nu X_2, (\nu a_1 - \mu a_2)] = 0. \quad (27)$$

当  $\mu = \nu$  时,以上结果就退化为文献[1]的结果.因此,我们就可以进一步计算带参数的双模光场正交分量  $(\mu X_1 + \nu X_2)/\lambda$  的函数的若干正规乘积展开式.

由(24)和(26)式我们可以得到

$$\begin{aligned} & \left[\frac{1}{\lambda}(\mu X_1 + \nu X_2)\right]^n \text{ 的正规乘积展开} \\ & \left[\frac{\mu X_1 + \nu X_2}{\lambda}\right]^n \\ &= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d^2\alpha}{2\pi} x^n |x, \alpha, \mu, \nu\rangle \langle x, \alpha|. \quad (28) \end{aligned}$$

对  $\int_{-\infty}^{\infty} d^2\alpha$  积分,利用(12)和(20)式,立即得到

$$\begin{aligned} & \left[\frac{1}{\lambda}(\mu X_1 + \nu X_2)\right]^n \\ &= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} x^n \\ & \quad \times \exp\left[-\left(x - \frac{\mu a_1^\dagger + \mu a_1 + \nu a_2^\dagger + \nu a_2}{\sqrt{2}\lambda}\right)^2\right] : \\ &= \sum_{k=0}^{n/2} \frac{n!}{2^{2k} k! (n-2k)!} \\ & \quad \times \left(\frac{\mu a_1^\dagger + \mu a_1 + \nu a_2^\dagger + \nu a_2}{\sqrt{2}\lambda}\right)^{n-2k} : \\ &= \frac{1}{(2i)^n} :H_n\left[i\frac{1}{\lambda}(\mu X_1 + \nu X_2)\right]:, \quad (29) \end{aligned}$$

此即其正规乘积展开式.类似地,利用上式我们可以

立即求出它的相干态期望值

$$\begin{aligned} & z_1, z_2 \left| \left[ \frac{\mu X_1 + \nu X_2}{\lambda} \right]^n \right|_{z_1, z_2} \\ &= \frac{1}{(2i)^n} H_n \left[ \frac{\sqrt{2}i}{\lambda} (\mu \operatorname{Re} z_1 + \nu \operatorname{Re} z_2) \right]. \quad (30) \end{aligned}$$

利用(24)和(26)式我们可以进一步得到

$\exp\left[\frac{i\gamma}{\lambda^2}(\mu X_1 + \nu X_2)^2\right]$  的正规乘积展开

$$\begin{aligned} & \exp\left[\frac{i\gamma}{\lambda^2}(\mu X_1 + \nu X_2)^2\right] \\ &= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d^2\alpha}{2\pi} \\ & \times \exp(i\gamma x^2) |x, \alpha, \mu, \nu, x, \alpha\rangle. \quad (31) \end{aligned}$$

用有序算符内的积分技术进行积分后得到

$$\begin{aligned} & \exp\left[\frac{i\gamma}{\lambda^2}(\mu X_1 + \nu X_2)^2\right] \\ &= \frac{1}{\sqrt{1-i\gamma}} \exp\left[\frac{i\gamma}{2\lambda^2(1-i\lambda)}(\mu a_1^\dagger + \nu a_2^\dagger)^2\right] \\ & \times \exp\left[\frac{i\gamma(\mu^2 a_1 a_1 + \nu a_2 a_2 + \mu\nu a_1 a_2 + \mu\nu a_2 a_1)}{\lambda^2(1-i\gamma)}\right. \\ & \left. - a_1^\dagger a_1 - a_2^\dagger a_2\right] : \\ & \times \exp\left[\frac{i\gamma}{2\lambda^2(1-i\lambda)}(\mu a_1 + \nu a_2)^2\right], \quad (32) \end{aligned}$$

把(32)式作用于双模真空态得到一个带参数的单-双模组合压缩态,即

$$\begin{aligned} & \exp\left[\frac{i\gamma}{\lambda^2}(\mu X_1 + \nu X_2)^2\right] |00\rangle \\ &= \frac{1}{\sqrt{1-i\gamma}} \exp\left[\frac{i\gamma(\mu a_1^\dagger + \nu a_2^\dagger)^2}{2\lambda^2(1-i\gamma)}\right] |00\rangle. \quad (33) \end{aligned}$$

类似地,利用(24)和(26)式及积分公式(20)我们可得到厄米多项式算符  $H_n\left[\frac{1}{\lambda}(\mu X_1 + \nu X_2)\right]$  的正规乘积展开

$$\begin{aligned} & H_n\left[\frac{1}{\lambda}(\mu X_1 + \nu X_2)\right] \\ &= 2^{n/2} \left[ \frac{1}{\lambda}(\mu X_1 + \nu X_2) \right]^n. \quad (34) \end{aligned}$$

## 6. 结 论

利用<sup>[1,10]</sup>中新引进的相干-纠缠态  $|x, \alpha, \mu, \nu\rangle$  和有序算符内的积分技术,我们给出了关于双模光场正交分量  $(X_1 + X_2)/\sqrt{2}$  及其参数化的正交分量  $(\mu X_1 + \nu X_2)/\lambda$  的函数的若干正规乘积展开式,这对于研究场的高阶压缩行为和构建广义压缩态是有用的.利用非线性克尔介质和光场的相互作用,可以用光学手段实现相干-纠缠态(包括多模和高模纠缠),因而借住相干-纠缠态的特殊性质,有望更好地解决量子同步通信问题<sup>[14]</sup>.

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# Deriving operator identities by two-mode coherent-entangled state representation \*

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## Abstract

On the basis of the new introduced two-mode coherent entangled state representation and the technique of integration within an ordered product of operators, we derived some operator identities of the operator functions about the quadrature of two-mode optical field, such as  $(X_1 + X_2)\sqrt{2}$  and  $(\mu X_1 + \nu X_2)\lambda$ , which should be helpful to further studying the high order squeezing properties of optical field and to constructing some generalized-squeezed states.

**Keywords** : two-mode coherent state, operator identity, the integration within an ordered product of operators technique

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