第 58 卷 第 6 期 2009 年 6 月
1000-3290/2009/58(06)/3625-07
物 理 学 报
ACTA PHYSICA SINICA
Vol.58, No.6, June, 2009
©2009 Chin. Phys. Soc.

离散差分变分 Hamilton 系统的 Lie 对称性与 Noether 守恒量*

施沈阳[1)†] 黄晓虹[2)] 张晓波[1)] 金 立[1)]

1)(浙江理工大学物理系，杭州 310018)

2)(温州大学物理与电子信息学院，温州 325000)

(2008 年 6 月 19 日收到；2008 年 7 月 11 日收到修改稿)

研究离散差分 Hamilton 系统的 Lie 对称性与 Noether 守恒量. 根据扩展的时间离散力学变分原理构建 Hamilton 系统的差分动力学方程. 定义离散系统运动差分方程在无限小变换群下的不变性为 Lie 对称性，导出由 Lie 对称性得到系统离散 Noether 守恒量的判据. 举例说明结果的应用.

关键词：离散力学，差分 Hamilton 系统，Lie 对称性，Noether 守恒量

PACC：0320，4610

1. 引 言

基于离散差分模型的约束力学系统的对称性与守恒量研究是近年来分析力学的研究方向之一，其主要思想是利用离散力学的变分原理把原来的连续系统离散为差分动力学方程形式的力学系统，从而构建起保原系统几何结构的离散算法[1,2]. Cadzow[3] 在研究离散系统的最优化问题时，提出了离散变分原理，并给出了离散运动方程. Lee[4,5] 首次提出将时间视作一个动力学变量和空间变量一起离散化，并给出了新的变分原理——差分变分原理. 这一离散变分思想被陈景波等[6,7] 扩展运用于 Lagrange 与 Hamilton 系统，得到了离散力学系统的 Euler-Lagrange 方程、正则方程与能量演化方程. 近年来，郭汉英等[8—10] 给出了离散力学一类新型变分形式——差分离散变分原理，并提出‘Euler-Lagrange 上同调’的概念. 张宏彬[11—14] 与傅景礼[15] 等研究了非保守非完整系统、事件空间系统、Hamilton 系统、Birkhoff 系统、机电系统等几类约束力学系统的离散变分原理和第一积分.

对称性理论在常微分方程的积分、偏微分方程的约化、线性化和渐进解以及分岔和控制等方面有广泛的运用，其中一个比较重要的用途就是用来寻求守恒量. 约束力学系统的对称性与守恒量研究近年来取得了很多重要的结果[16—20]. Lie 对称性和 Noether 对称性是近代寻求连续约束力学系统守恒量的两种常用的重要方法. Lie 对称性是运动微分方程在无限小变换群下的不变性[16,17,21—23]. Noether 对称性是基于 Hamilton 泛函作用量在无限小变换群下的不变性[16,17,24,25]. Lie 对称性和 Noether 对称性方法近来被推广运用到离散系统，其中，Levi 等[26—28] 从数学方面较详细地研究了离散差分方程和微分差分方程的 Lie 对称性理论，Dorodnitsyn[29] 建立了离散 Lagrange 系统的 Noether 理论. 本文在此基础上研究离散差分 Hamilton 系统的 Lie 对称性，并导出由 Lie 对称性得到离散 Noether 守恒量的条件.

2. 离散 Hamilton 系统的运动差分方程

力学系统的位形空间由流形 Q 来确定，其速度相空间由流形 Q 的切丛 TQ 来确定，考虑单自由度的位形流形用广义坐标 q 表示，速度相空间用广义动量 p 表示. 系统用 Lagrange 函数 $L(t, q, \dot{q})$ 或 Hamilton 函数 $H(t, q, p)$ 来表示，Lagrange 函数和 Hamilton 函数通过 Legendre 变换建立等价关系，有

$$\dot{q} = \frac{\partial H}{\partial p}, \tag{1}$$

* 国家自然科学基金(批准号：10672143)资助的课题.

† E-mail：hzshishenyang@sina.com

$$L(t,q,\dot{q}) = p\dot{q} - H(t,q,p), \tag{2}$$

系统的 Hamilton 泛函作用量表示为

$$\begin{aligned} S &= \int_{t_1}^{t_2} L(t,q,\dot{q})\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} [p\dot{q} - H(t,q,p)]\,\mathrm{d}t \\ &= \int_{q_1}^{q_2} p\,\mathrm{d}q - \int_{t_1}^{t_2} H(t,q,p)\,\mathrm{d}t. \end{aligned} \tag{3}$$

离散 Hamilton 力学中用离散差分序列$(q_i(t_i))$和$(p_i(t_i))$$(i=0,1,\cdots,N)$来替代连续的位形曲线$q(t)$和相空间中的连续曲线$p(t)$，离散形式的 Hamilton 函数和 Lagrange 函数分别表示为$H_\mathrm{d}(t_k, t_{k+1}, q_k, q_{k+1}, p_k, p_{k+1})$，$L_d(t_k, t_{k+1}, q_k, q_{k+1})$$(k=0,1,\cdots,N-1)$，其中的连续坐标$p$和$q$用离散中性差分坐标形式$(p_k+p_{k+1})/2$，$(q_k+q_{k+1})/2$表示，相应的离散形式 Hamilton 泛函作用量(3)式定义为

$$\begin{aligned} S_\mathrm{d} &= \sum_{k=0}^{N-1} \frac{1}{2}(p_{k+1}+p_k)(q_{k+1}-q_k) \\ &\quad - \sum_{k=0}^{N-1} H_\mathrm{d}(t_k, t_{k+1}, q_k, q_{k+1}, p_k, p_{k+1}) \\ &\quad \times (t_{k+1}-t_k). \end{aligned} \tag{4}$$

计算离散 Hamilton 作用量(4)式的全变分

$$\begin{aligned} \Delta S_\mathrm{d} &= \Delta \sum_{k=0}^{N-1} \frac{1}{2}(p_{k+1}+p_k)(q_{k+1}-q_k) \\ &\quad - \Delta \sum_{k=0}^{N-1} H_\mathrm{d}(\varphi_k)(t_{k+1}-t_k) \\ &= \sum_{k=1}^{N-1} [H_\mathrm{d}(\varphi_k) - H_\mathrm{d}(\varphi_{k-1}) \\ &\quad - D_1 H_\mathrm{d}(\varphi_k)(t_{k+1}-t_k) \\ &\quad - D_2 H_\mathrm{d}(\varphi_{k-1})(t_k - t_{k-1})]\Delta t_k \\ &\quad + \sum_{k=1}^{N-1} \Big[\frac{1}{2}(p_{k-1}-p_{k+1}) \\ &\quad - D_3 H_\mathrm{d}(\varphi_k)(t_{k+1}-t_k) \\ &\quad - D_4 H_\mathrm{d}(\varphi_{k-1})(t_k - t_{k-1})\Big]\Delta q_k \\ &\quad + \sum_{k=1}^{N-1} \Big[\frac{1}{2}(q_{k+1}-q_{k-1}) \\ &\quad - D_5 H_\mathrm{d}(\varphi_k)(t_{k+1}-t_k) \\ &\quad - D_6 H_\mathrm{d}(\varphi_{k-1})(t_k - t_{k-1})\Big]\Delta p_k \\ &\quad + [H_\mathrm{d}(\varphi_0) - D_1 H_\mathrm{d}(\varphi_0)(t_1-t_0)]\Delta t_0 \\ &\quad - [D_2 H_\mathrm{d}(\varphi_{N-1})(t_N - t_{N-1}) \\ &\quad + H_\mathrm{d}(\varphi_{N-1})]\Delta t_N \\ &\quad - \Big[\frac{1}{2}(p_1+p_0) + D_3 H_\mathrm{d}(\varphi_0)(t_1-t_0)\Big]\Delta q_0 \\ &\quad + \Big[\frac{1}{2}(p_N+p_{N-1}) \\ &\quad - D_4 H_\mathrm{d}(\varphi_{N-1})(t_N - t_{N-1})\Big]\Delta q_N \\ &\quad + \Big[\frac{1}{2}(q_1-q_0) - D_5 H_\mathrm{d}(\varphi_0)(t_1-t_0)\Big]\Delta p_0 \\ &\quad + \Big[\frac{1}{2}(q_N-q_{N-1}) \\ &\quad - D_6 H_\mathrm{d}(\varphi_{N-1})(t_N - t_{N-1})\Big]\Delta p_N, \end{aligned} \tag{5}$$

式中$\varphi_k = (t_k, t_{k+1}, q_k, q_{k+1}, p_k, p_{k+1})$$(k=0,1,\cdots,N-1)$，$D_j$表示对离散差分函数$H_\mathrm{d}$第$j$个变量的偏导数，结合固定端点条件$\Delta t_0 = \Delta t_N = 0$，$\Delta q_0 = \Delta q_N = 0$和$\Delta p_0 = \Delta p_N = 0$，根据扩展的时间离散力学变分原理$\Delta S_\mathrm{d} = 0$，对应于变分$\Delta q_k$和$\Delta p_k$，有

$$\begin{aligned} &\frac{1}{2}(p_{k-1}-p_{k+1}) \\ &- D_3 H_\mathrm{d}(\varphi_k)(t_{k+1}-t_k) \\ &- D_4 H_\mathrm{d}(\varphi_{k-1})(t_k - t_{k-1}) = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} &\frac{1}{2}(q_{k+1}-q_{k-1}) \\ &- D_5 H_\mathrm{d}(\varphi_k)(t_{k+1}-t_k) \\ &- D_6 H_\mathrm{d}(\varphi_{k-1})(t_k - t_{k-1}) = 0, \end{aligned} \tag{7}$$

等式(6)，(7)是离散 Hamilton 系统的差分正则方程，对应于变分Δt_k有

$$\begin{aligned} &H_\mathrm{d}(\varphi_k) - H_\mathrm{d}(\varphi_{k-1}) \\ &- D_1 H_\mathrm{d}(\varphi_k)(t_{k+1}-t_k) \\ &- D_2 H_\mathrm{d}(\varphi_{k-1})(t_k - t_{k-1}) = 0. \end{aligned} \tag{8}$$

等式(8)为离散 Hamilton 系统的能量演化方程，(6)—(8)式称为离散 Hamilton 系统的差分动力学方程.

3. 离散 Hamilton 系统的 Lie 对称性

取离散时间t_k、广义坐标q_k与广义动量p_k的无限小变换群

$$\begin{aligned} t_k^* &= t_k + \Delta t_k \\ &= t_k + \varepsilon\tau_k(t_k, q_k, p_k), \end{aligned} \tag{9}$$

$$\begin{aligned} q_k^* &= q_k + \Delta q_k \\ &= q_k + \varepsilon\xi_k(t_k, q_k, p_k), \end{aligned} \tag{10}$$

$$\begin{aligned} p_k^* &= p_k + \Delta p_k \\ &= p_k + \varepsilon\eta_k(t_k, q_k, p_k), \end{aligned} \tag{11}$$

(9)—(11)式中,ε 为群参数,τ_k,ξ_k,η_k 为变换群的离散生成元序列函数. 生成元的矢量场表示为

$$X_{\mathrm{d}}^{(0)}=\tau_k\frac{\partial}{\partial t_k}+\xi_k\frac{\partial}{\partial q_k}+\eta_k\frac{\partial}{\partial p_k},\tag{12}$$

矢量场(12)两个离散点与三个离散点的扩展图式分别表示为

$$X_{\mathrm{d}}^{(1)}=X_{\mathrm{d}}^{(0)}+\tau_{k+1}\frac{\partial}{\partial t_{k+1}}+\xi_{k+1}\frac{\partial}{\partial q_{k+1}}+\eta_{k+1}\frac{\partial}{\partial p_{k+1}},\tag{13}$$

$$X_{\mathrm{d}}^{(2)}=X_{\mathrm{d}}^{(1)}+\tau_{k-1}\frac{\partial}{\partial t_{k-1}}+\xi_{k-1}\frac{\partial}{\partial q_{k-1}}+\eta_{k-1}\frac{\partial}{\partial p_{k-1}},\tag{14}$$

把动力学方程(6)—(8)改写为离散差分方程的形式

$$\begin{aligned}&\frac{1}{2}(p_{k-1}-p_{k+1})\\&-D_3H_{\mathrm{d}}(\varphi_k)(t_{k+1}-t_k)\\&-D_4H_{\mathrm{d}}(\varphi_{k-1})(t_k-t_{k-1})\\=\;&U(t_{k-1},t_k,t_{k+1},q_{k-1},q_k,\\&q_{k+1},p_{k-1},p_k,p_{k+1})\\=\;&0,\end{aligned}\tag{15}$$

$$\begin{aligned}&\frac{1}{2}(q_{k+1}-q_{k-1})\\&-D_5H_{\mathrm{d}}(\varphi_k)(t_{k+1}-t_k)\\&-D_6H_{\mathrm{d}}(\varphi_{k-1})(t_k-t_{k-1})\\=\;&V(t_{k-1},t_k,t_{k+1},q_{k-1},q_k,\\&q_{k+1},p_{k-1},p_k,p_{k+1})\\=\;&0,\end{aligned}\tag{16}$$

$$\begin{aligned}&H_{\mathrm{d}}(\varphi_k)-H_{\mathrm{d}}(\varphi_{k-1})\\&-D_1H_{\mathrm{d}}(\varphi_k)(t_{k+1}-t_k)\\&-D_2H_{\mathrm{d}}(\varphi_{k-1})(t_k-t_{k-1})\\=\;&W(t_{k-1},t_k,t_{k+1},q_{k-1},q_k,\\&q_{k+1},p_{k-1},p_k,p_{k+1})\\=\;&0,\end{aligned}\tag{17}$$

其中

$$\begin{aligned}\omega_k=\;&(t_{k-1},t_k,t_{k+1},q_{k-1},q_k,\\&q_{k+1},p_{k-1},p_k,p_{k+1})\\&(k=1,\cdots,N-1).\end{aligned}$$

运用(14)式于离散差分方程(15)—(17),得到

$$X_{\mathrm{d}}^{(2)}[U(\omega_k)]=0,\tag{18}$$

$$X_{\mathrm{d}}^{(2)}[V(\omega_k)]=0,\tag{19}$$

$$X_{\mathrm{d}}^{(2)}[W(\omega_k)]=0.\tag{20}$$

判据 1　如果离散生成元序列函数 τ_k,ξ_k,η_k 满足方程(18)—(20),则相应的离散差分动力学方程(6)—(8)的不变性为离散差分 Hamilton 系统的 Lie 对称性,方程(18)—(20)为 Lie 对称性的确定方程.

4. 离散 Hamilton 系统的 Noether 守恒量

引入离散变量和离散函数的递推算符和一次导数算符分别为

$$R_{\pm}f(z_k)=f(z_{k\pm1}),\tag{21}$$

$$D_{\mathrm{d}}f(z_k)=\frac{R_+f(z_k)-f(z_k)}{t_{k+1}-t_k}.\tag{22}$$

离散系统的 Noether 对称性是指差分形式的 Hamilton 泛函作用量在无限小变换群下的不变性,Noether 对称性总能得到相应的 Noether 守恒量. 离散差分 Hamilton 系统的 Noether 对称性有如下判据:

判据 2　如果离散 Hamilton 系统的差分动力学方程(6)—(8)成立,且存在离散规范函数 $G_{Nk}(t_k,q_k,p_k)$满足下列等式

$$\begin{aligned}&H_{\mathrm{d}}(\varphi_k)D_{\mathrm{d}}(\tau_k)+X_{\mathrm{d}}^{(1)}[H_{\mathrm{d}}(\varphi_k)]\\&-\frac{1}{2}(p_{k+1}+p_k)D_{\mathrm{d}}(\xi_k)\\&+\frac{1}{2}(q_{k+1}+q_k)D_{\mathrm{d}}(\eta_k)\\&+D_{\mathrm{d}}(G_{Nk})=0,\end{aligned}\tag{23}$$

则系统存在如下形式的离散 Noether 守恒量

$$\begin{aligned}&\tau_k(t_k-t_{k-1})D_2[R_-H_{\mathrm{d}}(\varphi_k)]\\&+\xi_k(t_k-t_{k-1})D_4[R_-H_{\mathrm{d}}(\varphi_k)]\\&+\eta_k(t_k-t_{k-1})D_6[R_-H_{\mathrm{d}}(\varphi_k)]\\&+\tau_kR_-H_{\mathrm{d}}(\varphi_k)\\&-\frac{1}{2}(p_{k-1}+p_k)\xi_k\\&+\frac{1}{2}(q_{k-1}+q_k)\eta_k+G_{Nk}\\=\;&\mathrm{const}.\end{aligned}\tag{24}$$

(23)式称为 Hamilton 形式的离散差分形式的 Noether 恒等式.

证明　根据(12),(13),(21),(22)式展开等式(23)得到

$$\begin{aligned}&H_{\mathrm{d}}(\varphi_k)D_{\mathrm{d}}(\tau_k)+X_{\mathrm{d}}^{(1)}[H_{\mathrm{d}}(\varphi_k)]\\&-\frac{1}{2}(p_{k+1}+p_k)D_{\mathrm{d}}(\xi_k)\\&+\frac{1}{2}(q_{k+1}+q_k)D_{\mathrm{d}}(\eta_k)+D_{\mathrm{d}}(G_{Nk})\end{aligned}$$

$$
\begin{aligned}
&= \tau_k\left[\frac{\partial H_{\mathrm{d}}(\varphi_k)}{\partial t_k} + \frac{t_k - t_{k-1}}{t_{k+1} - t_k}\frac{\partial H_{\mathrm{d}}(\varphi_{k-1})}{\partial t_k}\right. \\
&\quad\left. + \frac{H_{\mathrm{d}}(\varphi_{k-1}) - H_{\mathrm{d}}(\varphi_k)}{t_{k+1} - t_k}\right] \\
&\quad + \xi_k\left[\frac{\partial H_{\mathrm{d}}(\varphi_k)}{\partial q_k} + \frac{t_k - t_{k-1}}{t_{k+1} - t_k}\frac{\partial H_{\mathrm{d}}(\varphi_{k-1})}{\partial q_k}\right. \\
&\quad\left. - \frac{1}{2}\frac{p_{k-1} - p_{k+1}}{t_{k+1} - t_k}\right] \\
&\quad + \eta_k\left[\frac{\partial H_{\mathrm{d}}(\varphi_k)}{\partial p_k} + \frac{t_k - t_{k-1}}{t_{k+1} - t_k}\frac{\partial H_{\mathrm{d}}(\varphi_{k-1})}{\partial p_k}\right. \\
&\quad\left. - \frac{1}{2}\frac{q_{k+1} - q_{k-1}}{t_{k+1} - t_k}\right] \\
&\quad + D_{\mathrm{d}}\left[\tau_k(t_k - t_{k-1})\frac{\partial H_{\mathrm{d}}(\varphi_{k-1})}{\partial t_k}\right. \\
&\quad + \tau_k(t_k - t_{k-1})\frac{\partial H_{\mathrm{d}}(\varphi_{k-1})}{\partial q_k} \\
&\quad + \tau_k(t_k - t_{k-1})\frac{\partial H_{\mathrm{d}}(\varphi_{k-1})}{\partial p_k} \\
&\quad + \tau_k H_{\mathrm{d}}(\varphi_{k-1}) - \frac{1}{2}(p_{k-1} + p_k)\xi_k \\
&\quad\left. + \frac{1}{2}(q_{k-1} + q_k)\eta_k + G_{Nk}\right] \\
&= 0,
\end{aligned} \tag{25}
$$

结合差分动力学方程(6)—(8),有

$$
\begin{aligned}
&D_{\mathrm{d}}\left[\tau_k(t_k - t_{k-1})\frac{\partial H_{\mathrm{d}}(\varphi_{k-1})}{\partial t_k}\right. \\
&\quad + \tau_k(t_k - t_{k-1})\frac{\partial H_{\mathrm{d}}(\varphi_{k-1})}{\partial q_k} \\
&\quad + \tau_k(t_k - t_{k-1})\frac{\partial H_{\mathrm{d}}(\varphi_{k-1})}{\partial p_k} \\
&\quad + \tau_k H_{\mathrm{d}}(\varphi_{k-1}) - \frac{1}{2}(p_{k-1} + p_k)\xi_k \\
&\quad\left. + \frac{1}{2}(q_{k-1} + q_k)\eta_k + G_{Nk}\right] \\
&= 0.
\end{aligned} \tag{26}
$$

根据(21),(22)式,由(26)式可得离散 Noether 守恒量(24)

$$
\begin{aligned}
&\tau_k(t_k - t_{k-1})D_2[R_- H_{\mathrm{d}}(\varphi_k)] \\
&\quad + \xi_k(t_k - t_{k-1})D_4[R_- H_{\mathrm{d}}(\varphi_k)] \\
&\quad + \eta_k(t_k - t_{k-1})D_6[R_- H_{\mathrm{d}}(\varphi_k)] \\
&\quad + \tau_k R_- H_{\mathrm{d}}(\varphi_k) - \frac{1}{2}(p_{k-1} + p_k)\xi_k \\
&\quad + \frac{1}{2}(q_{k-1} + q_k)\eta_k + G_{Nk} \\
&= \mathrm{const}.
\end{aligned} \tag{27}
$$

根据判据 1 和判据 2 可直接得到由 Lie 对称性得到 Noether 守恒量的条件:

判据 3 如果离散差分 Hamilton 系统 Lie 对称性的生成元序列函数 τ_k,ξ_k,η_k 满足 Noether 对称性的判据方程(23),则离散系统 Lie 对称性导致 Noether 守恒量(24).

由判据 1 和判据 2 可直接证明判据 3.

5. 算 例

离散差分 Hamilton 形式 Emden 方程的 Hamilton 函数为

$$
\begin{aligned}
H_{\mathrm{d}}(\varphi_k) &= \frac{1}{2}\frac{p_{k+1}^2 + p_k^2}{t_{k+1}^2 + t_k^2} \\
&\quad + \frac{1}{2}\left(\frac{1}{6}t_{k+1}^2 q_{k+1}^6 + \frac{1}{6}t_k^2 q_k^6\right),
\end{aligned} \tag{28}
$$

研究其 Lie 对称性与 Noether 守恒量.

递推形式的 Hamilton 函数为

$$
\begin{aligned}
H_{\mathrm{d}}(\varphi_{k-1}) &= \frac{1}{2}\frac{p_k^2 + p_{k-1}^2}{t_k^2 + t_{k-1}^2} \\
&\quad + \frac{1}{12}(t_k^2 q_k^6 + t_{k-1}^2 q_{k-1}^6),
\end{aligned} \tag{29}
$$

(28),(29)式代入差分动力学方程(6)—(8),并计算 $D_1 H_{\mathrm{d}}(\varphi_k)$, $D_3 H_{\mathrm{d}}(\varphi_k)$, $D_5 H_{\mathrm{d}}(\varphi_k)$, $D_2 H_{\mathrm{d}}(\varphi_{k-1})$, $D_4 H_{\mathrm{d}}(\varphi_{k-1})$, $D_6 H_{\mathrm{d}}(\varphi_{k-1})$,得到系统的离散差分正则方程和能量演化方程

$$
\begin{aligned}
&U(t_{k-1}, t_k, t_{k+1}, q_{k-1}, q_k, \\
&\quad q_{k+1}, p_{k-1}, p_k, p_{k+1}) \\
&= \frac{1}{2}(p_{k-1} - p_{k+1}) \\
&\quad - \frac{1}{2}t_k^2 q_k^5(t_{k+1} - t_{k-1}) \\
&= 0,
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
&V(t_{k-1}, t_k, t_{k+1}, q_{k-1}, q_k, \\
&\quad q_{k+1}, p_{k-1}, p_k, p_{k+1}) \\
&= \frac{1}{2}(q_{k+1} - q_{k-1}) \\
&\quad - p_k\frac{t_{k+1} - t_k}{t_{k+1}^2 + t_k^2} \\
&\quad - p_k\frac{t_k - t_{k-1}}{t_k^2 + t_{k-1}^2} \\
&= 0,
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
&W(t_{k-1}, t_k, t_{k+1}, q_{k-1}, q_k, \\
&\quad q_{k+1}, p_{k-1}, p_k, p_{k+1})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\frac{p_{k+1}^2 + p_k^2}{t_{k+1}^2 + t_k^2} - \frac{p_k^2 + p_{k-1}^2}{t_k^2 + t_{k-1}^2}\right) \\
&\quad + \frac{1}{2}\left(\frac{1}{6}t_{k+1}^2 q_{k+1}^6 - \frac{1}{6}t_{k-1}^2 q_{k-1}^6\right) \\
&\quad + (p_{k+1}^2 + p_k^2)\frac{t_k(t_{k+1} - t_k)}{(t_{k+1}^2 + t_k^2)^2} \\
&\quad + (p_k^2 + p_{k-1}^2)\frac{t_k(t_k - t_{k-1})}{(t_k^2 + t_{k-1}^2)^2} \\
&\quad - \frac{1}{6}t_k q_k^6 (t_{k+1} - t_{k-1}) = 0.
\end{aligned} \tag{32}
$$

(30)—(32)式代入(18)—(20)式得到 Lie 对称性的确定方程

$$
\begin{aligned}
&\tau_{k-1}D_1 U(\omega_k) + \tau_k D_2 U(\omega_k) \\
&+ \tau_{k+1}D_3 U(\omega_k) + \xi_{k-1}D_4 U(\omega_k) \\
&+ \xi_k D_5 U(\omega_k) + \xi_{k+1}D_6 U(\omega_k) \\
&+ \eta_{k-1}D_7 U(\omega_k) + \eta_k D_8 U(\omega_k) \\
&+ \eta_{k+1}D_9 U(\omega_k) \\
&= 0,
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
&\tau_{k-1}D_1 V(\omega_k) + \tau_k D_2 V(\omega_k) \\
&+ \tau_{k+1}D_3 V(\omega_k) + \xi_{k-1}D_4 V(\omega_k) \\
&+ \xi_k D_5 V(\omega_k) + \xi_{k+1}D_6 V(\omega_k) \\
&+ \eta_{k-1}D_7 V(\omega_k) + \eta_k D_8 V(\omega_k) \\
&+ \eta_{k+1}D_9 V(\omega_k) \\
&= 0,
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
&\tau_{k-1}D_1 W(\omega_k) + \tau_k D_2 W(\omega_k) \\
&+ \tau_{k+1}D_3 W(\omega_k) + \xi_{k-1}D_4 W(\omega_k) \\
&+ \xi_k D_5 W(\omega_k) + \xi_{k+1}D_6 W(\omega_k) \\
&+ \eta_{k-1}D_7 W(\omega_k) + \eta_k D_8 W(\omega_k) \\
&+ \eta_{k+1}D_9 W(\omega_k) \\
&= 0.
\end{aligned} \tag{35}
$$

设离散生成元序列函数有如下形式：

$$
\begin{aligned}
&\tau_k(t_k, q_k, p_k) \\
&= C_1 t_k + C_2 q_k + C_3 p_k + C_4,
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
&\xi_k(t_k, q_k, p_k) \\
&= C_5 t_k + C_6 q_k + C_7 p_k + C_8
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
&\eta_k(t_k, q_k, p_k) \\
&= C_9 t_k + C_{10} q_k + C_{11} p_k + C_{12},
\end{aligned} \tag{38}
$$

式中 C_1—C_{12} 为常数，(36)—(38)式代入确定方程(33)，并计算 $D_{1-9}U(\omega_k)$，比较各项的系数可知，当

$$C_2 = C_3 = C_4 = C_6 = C_7 = C_8 = 0, \tag{39}$$

$$C_9 = 1, C_1 = 2C_9, C_5 = -C_9, \tag{40}$$

即

$$\tau_k(t_k, q_k, p_k) = 2t_k, \tag{41}$$

$$\xi_k(t_k, q_k, p_k) = -q_k, \tag{42}$$

$$\eta_k(t_k, q_k, p_k) = p_k \tag{43}$$

时，有

$$
\begin{aligned}
X_{\mathrm{d}}^{(2)}[U(\omega_k)] &= \frac{1}{2}t_k^2 q_k^5 (t_{k+1} - t_{k-1}) \\
&\quad + \frac{1}{2}(p_{k+1} - p_{k-1}).
\end{aligned} \tag{44}
$$

结合(30)式，(44)式化为

$$X_{\mathrm{d}}^{(2)}[U(\omega_k)] = 0. \tag{45}$$

利用同样的方法计算 $D_{1-9}V(\omega_k)$，$D_{1-9}W(\omega_k)$，并把(41)—(43)式代入确定方程(34)，(35)，得到

$$
\begin{aligned}
X_{\mathrm{d}}^{(2)}[V(\omega_k)] &= p_k\frac{t_{k+1} - t_k}{t_{k+1}^2 + t_k^2} + p_k\frac{t_k - t_{k-1}}{t_k^2 + t_{k-1}^2} \\
&\quad - \frac{1}{2}(q_{k+1} - q_{k-1}),
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
X_{\mathrm{d}}^{(2)}[W(\omega_k)] &= \frac{p_k^2 + p_{k-1}^2}{t_k^2 + t_{k-1}^2} - \frac{p_{k+1}^2 + p_k^2}{t_{k+1}^2 + t_k^2} \\
&\quad + \frac{1}{6}t_{k-1}^2 q_{k-1}^6 - \frac{1}{6}t_{k+1}^2 q_{k+1}^6 \\
&\quad - 2(p_{k+1}^2 + p_k^2)\frac{t_k(t_{k+1} - t_k)}{(t_{k+1}^2 + t_k^2)^2} \\
&\quad - 2(p_k^2 + p_{k-1}^2)\frac{t_k(t_k - t_{k-1})}{(t_k^2 + t_{k-1}^2)^2} \\
&\quad + \frac{1}{3}t_k q_k^6 (t_{k+1} - t_{k-1}).
\end{aligned} \tag{47}
$$

结合(31)，(32)式，(46)，(47)式化为

$$X_{\mathrm{d}}^{(2)}[V(\omega_k)] = -V(\omega_k) = 0, \tag{48}$$

$$X_{\mathrm{d}}^{(2)}[W(\omega_k)] = -2W(\omega_k) = 0. \tag{49}$$

由此可知离散生成元函数(41)—(43)为离散 Hamilton 形式 Emden 方程的 Lie 对称性生成元.

计算 $D_j[H_{\mathrm{d}}(\varphi_k)]$ $(j = 1,2,3,4,5,6)$，并结合(28)式，代入离散差分 Noether 等式(23)得到

$$
\begin{aligned}
&\left[\frac{1}{2}\frac{p_{k+1}^2 + p_k^2}{t_{k+1}^2 + t_k^2} + \frac{1}{12}(t_{k+1}^2 q_{k+1}^6 + t_k^2 q_k^6)\right]D_{\mathrm{d}}(\tau_k) \\
&+ \left[\frac{1}{6}t_k q_k^6 - \frac{t_k(p_{k+1}^2 + p_k^2)}{(t_{k+1}^2 + t_k^2)^2}\right]\tau_k \\
&+ \left[\frac{1}{6}t_{k+1} q_{k+1}^6 - \frac{t_{k+1}(p_{k+1}^2 + p_k^2)}{(t_{k+1}^2 + t_k^2)^2}\right]\tau_{k+1} \\
&+ \frac{1}{2}t_k^2 q_k^5 \xi_k + \frac{1}{2}t_{k+1}^2 q_{k+1}^5 \xi_{k+1} \\
&+ \frac{p_k \eta_k}{t_{k+1}^2 + t_k^2} + \frac{p_{k+1}\eta_{k+1}}{t_{k+1}^2 + t_k^2}
\end{aligned}
$$

$$
-\frac{1}{2}(p_{k+1}+p_k)D_{\mathrm{d}}(\xi_k)
$$

$$
+\frac{1}{2}(q_{k+1}+q_k)D_{\mathrm{d}}(\eta_k)+D_{\mathrm{d}}(G_{Nk})
$$

$$
=0. \tag{50}
$$

把离散生成元函数(41)—(43)代入(50)式,得到

$$
H_{\mathrm{d}}(\varphi_k)D_{\mathrm{d}}(\tau_k)+X_{\mathrm{d}}^{(1)}[H_{\mathrm{d}}(\varphi_k)]=0, \tag{51}
$$

并存在离散规范函数 $G_{Nk}(t_k,q_k,p_k)$,

$$
G_{Nk}(t_k,q_k,p_k)=-q_kp_k, \tag{52}
$$

使得 Noether 等式(23)成立,由此可知离散生成元函数(41)—(43)也是离散 Noether 对称性的生成元,根据判据 3 得到离散差分 Hamilton 系统的 Noether 守恒量

$$
\begin{aligned}
I_N &= \frac{t_k(p_k^2+p_{k-1}^2)}{t_k^2+t_{k-1}^2} \\
&\quad -\frac{2t_k^2(t_k-t_{k-1})(p_k^2+p_{k-1}^2)}{(t_k^2+t_{k-1}^2)^2} \\
&\quad +\frac{(t_k-t_{k-1})p_k^2}{t_k^2+t_{k-1}^2} \\
&\quad +\frac{1}{6}t_kt_{k-1}(t_kq_k^6+t_{k-1}q_{k-1}^6) \\
&\quad +\frac{1}{2}(p_{k-1}q_k+p_kq_{k-1}),
\end{aligned} \tag{53}
$$

其中由(29)式可计算 Noether 守恒量(24)中的 $D_2[H_{\mathrm{d}}(\varphi_{k-1})]$, $D_4[H_{\mathrm{d}}(\varphi_{k-1})]$, $D_6[H_{\mathrm{d}}(\varphi_{k-1})]$.

6. 结 论

本文研究离散差分 Hamilton 系统的 Lie 对称性、Noether 对称性、Lie 对称性导致的 Noether 守恒量.主要的结论有:离散差分 Hamilton 系统 Lie 对称性的确定方程(18)—(20);Noether 对称性的判据方程(23)与 Noether 守恒量(24);离散差分 Hamilton 形式 Emden 方程的 Lie 对称性与 Noether 守恒量.

[1] Wendlandt J M, Marsden J E 1997 *Physica* D **106** 223

[2] Marsden J E, West M 2001 *Acta Numerica* 357

[3] Cadzow J D 1970 *Int. J. Control* **11** 393

[4] Lee T D 1983 *Phys. Lett.* B **122** 217

[5] Lee T D 1987 *J. Stat. Phys.* **46** 843

[6] Chen J B, Guo H Y, Wu K 2006 *Appl. Math. Comput.* **177** 226

[7] Chen J B, Guo H Y, Wu K 2003 *J. Math. Phys.* **44** 1688

[8] Guo H Y, Li Y Q, Wu K, Wang S K 2002 *Commun. Theor. Phys.* (*Beijing*, *China*) **37** 1

[9] Guo H Y, Li Y Q, Wu K, Wang S K 2002 *Commun. Theor. Phys.* (*Beijing*, *China*) **37** 129

[10] Guo H Y, Li Y Q, Wu K, Wang S K 2002 *Commun. Theor. Phys.* (*Beijing*, *China*) **37** 257

[11] Zhang H B, Chen L Q, Liu R W 2005 *Chin. Phys.* **14** 238

[12] Zhang H B, Chen L Q, Liu R W 2005 *Chin. Phys.* **14** 888

[13] Zhang H B, Chen L Q, Liu R W 2005 *Chin. Phys.* **14** 1063

[14] Zhang H B, Chen L Q, Gu S L, Liu C Z 2007 *Chin. Phys.* **16** 582

[15] Fu J L, Dai G D, Jimenez S, Tang Y F 2007 *Chin. Phys.* **16** 570

[16] Mei F X 1999 *Applications of Lie groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese) [梅凤翔 1999 李群和李代数对约束力学系统的应用(北京:科学出版社)]

[17] Mei F X 2004 *Symmetries and Conserved Quantities of Constrained Mechanical Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese) [梅凤翔 2004 约束力学系统的对称性与守恒量(北京:北京理工大学出版社)]

[18] Shang M, Guo Y X, Mei F X 2007 *Chin. Phys.* **16** 292

[19] Lou Z M 2007 *Acta Phys. Sin.* **56** 2475 (in Chinese) [楼智美 2007 物理学报 **56** 2475]

[20] Mei F X, Xie J F, Gang T Q 2007 *Acta Phys. Sin.* **56** 5041 (in Chinese) [梅凤翔、解加芳、 铁强 2007 物理学报 **56** 5041]

[21] Jing H X, Li Y C, Xia L L 2007 *Acta Phys. Sin.* **56** 3043 (in Chinese) [荆宏星、李元成、夏丽丽 2007 物理学报 **56** 3043]

[22] Zhang Y 2007 *Acta Phys. Sin.* **56** 3054 (in Chinese) [张 毅 2007 物理学报 **56** 3054]

[23] Hu C L 2007 *Acta Phys. Sin.* **56** 3675 (in Chinese) [胡楚勒 2007 物理学报 **56** 3675]

[24] Wu H B, Mei F X 2006 *Acta Phys. Sin.* **55** 3825 (in Chinese) [吴惠彬、梅凤翔 2006 物理学报 **55** 3825]

[25] Shang M, Chen X W 2006 *Chin. Phys.* **15** 2788

[26] Levi D, Tremblay S, Winternitz P 2000 *J. Phys.* A: *Math. Gen.* **33** 8507

[27] Levi D, Yamilov R 1997 *J. Math. Phys.* **38** 6648

[28] Levi D, Tremblay S, Winternitz P 2001 *J. Phys.* A: *Math. Gen.* **34** 9507

[29] Dorodnitsyn V 2001 *Appl. Numer. Math.* **39** 307

The Lie symmetry and Noether conserved quantity of discrete difference variational Hamilton system *

Shi Shen-Yang[1)†] Huang Xiao-Hong[2)] Zhang Xiao-Bo[1)] Jin Li[1)]

1)(*Department of Physics*, *Zhejiang Sci-Tech University*, *Hangzhou* 310018, *China*)

2)(*School of Physics and Electronic Information*, *Wenzhou University*, *Wenzhou* 325000, *China*)

(Received 19 June 2008; revised manuscript received 11 July 2008)

Abstract

The Lie symmetry and Noether conserved quantity of discrete difference Hamilton system are investigated. Based on the extended mechanical variational principle of discrete time, the difference dynamical equations of Hamilton system are constructed. The invariance of difference equations of discrete system under infinitesimal transformation groups is defined to be Lie symmetry and the criterion for when discrete Noether conserved quantities may be obtained from Lie symmetries is also deduced. An example is discussed to show the applications of the results.

Keywords: discrete mechanics, difference Hamilton system, Lie symmetry, Noether conserved quantity

PACC: 0320, 4610

* Project supported by the National Natural Science Foundation of China (Grant No. 10672143).

† E-mail: hzshishenyang@sina.com