外加直流电场作用下高阶弱非线性复合介质的电势分布*

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利用同伦分析方法, 研究了一类由柱形杂质随机嵌入基质所形成的、电场和电流密度满足 $\boldsymbol{J}=\sigma\boldsymbol{E}+\chi|\boldsymbol{E}|^2\boldsymbol{E}+\eta|\boldsymbol{E}|^4\boldsymbol{E}$ 形式本构关系的高阶弱非线性复合介质在外加直流电场作用下的电势分布问题. 首先利用模函数展开法, 将本构方程及边界条件化成了一系列非线性常微分方程的边值问题; 再利用同伦分析方法进行求解, 给出了电势在基质和杂质区域的渐近解析解.

关键词: 高阶弱非线性复合介质, 模函数展开法, 同伦分析方法, 电势分布

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1 引 言

复合材料是当今物理学的一个研究热点, 而复合介质又是复合材料中最重要的一种, 研究其在外加电场作用下的电极化规律与介质结构的关系, 揭示其宏观介电性质的微观机制, 有利于充分发挥介质材料的效用并制造一些具有新型功能的复合介质材料 [1]. 复合介质是由两种或两种以上电导率不同的材料掺杂而成的, 一般由一种或几种杂质随机嵌入基质而形成, 具有宏观尺度上的不均匀性, 它不仅能保持其各组分材料的特性, 还增加了单一组分材料所不具有的综合性能 [1]. 对于这类复合介质, 在外加电场作用下, 其介电常数是综合反映其非线性输运性质最基本的宏观物理量, 其结果有助于分析此类复合介质材料的性能, 以期通过改变材料结构进而设计具有特殊性能的复合介质材料 [1], 因而对这种体系的研究既具重要的理论意义又有极大的实用价值.

自 20 世纪 80 年代以来 [2,3], 组分具有非线性的电位移矢量 ($\boldsymbol{D}$)- 电场 ($\boldsymbol{E}$) 关系 (如 $\boldsymbol{D}=\varepsilon\boldsymbol{E}+\chi|\boldsymbol{E}|^2\boldsymbol{E}$) 的颗粒复合介质的有效介电响应一直是数学物理学界广泛研究的问题之一. 在理论上, 通常采用谱表示法 [4−8]、T- 矩阵法 [9,10] 和微扰展开法 [11,12] 等研究复合介质材料的有效非线性响应. 这些方法各有特点 [10], 谱表示方法通过引入反映微结构的谱密度函数, 将表征微结构信息的参数和物质参数分离, 因此只要获得体系的谱密度函数, 就可以计算出体系的有效线性响应和三阶非线性极化率; T 矩阵法是将体系的有效非线性响应表示为一系列矩阵元的形式, 通过求解体系的矩阵元, 可以解析地给出复合体系的有效非线性响应; 对于微扰展开法, 由于弱非线性响应部分可以看作线性部分的微扰, 可通过求解非线性微分方程及静电边界条件, 从而获得非线性响应的解析表达式, 该方法可以用来处理颗粒杂质与基质均为非线性的情况. 近年来, 人们对在外加直流电场 [11−15] 或交流电场 [16−30] 或交直流混合电场 [31−36] 作用下非线性复合介质的有效介电响应问题进行了大量的研究, 得到了一系列的结论. 其中, 在外加直流电场作

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用下非线性复合介质的有效介电响应问题最为简单, 对其研究方法稍做改进即可用来研究在外加交流电场或交直流混合电场作用下非线性复合介质的有效介电响应问题, 因而具有本质上的重要性. 例如, 在外加直流电场作用的情形下, 文献 [11, 12] 利用微扰展开法导出了非线性复合介质的有效电导率; 文献 [13] 利用这种方法得出了球形杂质随机嵌入基质中所构成复合介质的有效电导率; 文献 [14] 给出了非线性复合介质的高阶有效性响应; 文献 [15] 得到了 Kerr 型非线性复合介质有效的直流电响应等等.

以上研究结果大多数是基于低阶弱非线性复合介质. 最近, 在外加电场作用下, 高阶弱非线性复合介质的介电特性逐渐引起人们的研究兴趣. 对于这类非线性复合介质, 在外加电场作用下的电势分布可以给出其介电性质的详细描述, 通过它可以计算电场分布、有效介电常数等相关的物理量, 通常可利用研究低阶弱非线性复合介质的微扰展开法来进行研究. 例如, Natenapit 等 [33] 利用微扰展开法研究了在外加交直流混合电场作用下一类高阶弱非线性复合介质的高阶响应等. 此类方法的优点是简单, 缺点是问题中必须要含有一个小参数. 随着求解强非线性问题新方法的不断出现, 已有一些学者开始利用数学上一些处理强非线性问题的方法来研究非线性复合介质在外加电场作用下的电势分布问题. 如 Wei 和 Gu[37] 利用同伦分析方法 [38] 研究了外加直流电场下一类低阶弱非线性复合介质的电势分布等等. 此类方法的优点是不依赖于小参数, 可推广用于处理一些强非线性复合介质的介电响应问题. 本文利用同伦分析方法 [37,38] 研究一类由柱形杂质随机嵌入基质所形成的、电场和电流密度满足 $\boldsymbol{J}=\sigma\boldsymbol{E}+\chi|\boldsymbol{E}|^2\boldsymbol{E}+\eta|\boldsymbol{E}|^4\boldsymbol{E}$ 形式本构关系的高阶弱非线性复合介质在外加直流电场作用下的电势分布问题.

2 方程和边界条件

考虑一个半径为 $r=a$ 的圆柱形杂质颗粒随机嵌入基质所形成的非线性复合介质, 外加电场为 $\boldsymbol{E}_a=\boldsymbol{E}_0$, 沿 $\hat{z}$ 轴正向作用于非线性复合介质.

假设电流密度 $\boldsymbol{J}$ 和电场强度 $\boldsymbol{E}$ 之间满足的本构关系为

$$\begin{aligned}\boldsymbol{J}=&\sigma\boldsymbol{E}+\chi|\boldsymbol{E}|^2\boldsymbol{E}+\eta|\boldsymbol{E}|^4\boldsymbol{E},\\ &\text{in }\Omega_\alpha,\quad \alpha=i,m,\end{aligned}\tag{1}$$

其中下标 $\alpha=i$, m 分别用来标记杂质和基质, Ω_i 和 Ω_m 分别表示颗粒杂质和基质所占空间区域, σ_α, χ_α 和 η_α 分别是线性电导率, 三阶和五阶非线性极化率, 且与外加电场无关.

电流密度 $\boldsymbol{J}$ 和电场强度 $\boldsymbol{E}$ 分别满足静态方程:

$$\nabla\cdot\boldsymbol{J}=0,\quad \nabla\times\boldsymbol{E}=0,\tag{2}$$

其边界条件为电势和电流密度在杂质和基质界面上是连续的, 即

$$\begin{aligned}&\varphi^m=\varphi^i\ \text{ on }\ \partial\Omega_i,\\ &\boldsymbol{n}\cdot\boldsymbol{J}^m=\boldsymbol{n}\cdot\boldsymbol{J}^i\ \text{ on }\ \partial\Omega_i,\end{aligned}\tag{3}$$

这里, 上标 m 和 i 分别记为在基质区和杂质区的量, $\partial\Omega_i$ 为杂质的表面.

3 模函数展开法

由于电场强度满足方程 (2), 故存在电势函数 $\varphi(x)$, 满足:

$$\boldsymbol{E}_a=-\nabla\varphi(x).\tag{4}$$

利用模函数展开法 [37], 令电势在基质和杂质区域作如下展开

$$\begin{aligned}\varphi^\alpha(r,\theta)=&\varphi_1^\alpha(r)\cos\theta+\varphi_3^\alpha(r)\cos3\theta+\cdots,\\ &\alpha=m,i,\end{aligned}\tag{5}$$

其中 $\varphi_n^\alpha(r)$ 叫做模函数.

将 (5) 式代入 (4) 式, 再代入 (1) 和 (2) 式及边界条件 (3) 式, 比较三角函数的系数可得到一系列的常微分方程的边值问题. 为方便起见, 我们保留到 $\cos3\theta$ 的项, 即

$$\varphi^\alpha(r,\theta)=\varphi_1^\alpha(r)\cos\theta+\varphi_3^\alpha(r)\cos3\theta,$$

代入 (4), (1), (3) 式, 则在基质区和杂质区 $\varphi_1^\alpha(r)$ 和 $\varphi_3^\alpha(r)$ 满足如下方程

$$\sigma_\alpha F_1[\varphi_1^\alpha]+\chi_\alpha G_1[\varphi_1^\alpha;\varphi_3^\alpha]+\eta_\alpha H_1[\varphi_1^\alpha;\varphi_3^\alpha]=0\quad \alpha=m,i\ \text{ in }\ \Omega_\alpha,\tag{6}$$

$$\varphi_1^i(r)=\varphi_1^m(r)\big|_{r=a},\tag{7}$$

$$\sigma_i J_1[\varphi_1^i]+\chi_i K_1[\varphi_1^i;\varphi_3^i]+\eta_i L_1[\varphi_1^i;\varphi_3^i]=\sigma_m J_1[\varphi_1^m]+\chi_m K_1[\varphi_1^m;\varphi_3^m]+\eta_m L_1[\varphi_1^m;\varphi_3^m]\big|_{r=a}, \tag{8}$$

$$\sigma_\alpha F_3[\varphi_3^\alpha]+\chi_\alpha G_3[\varphi_1^\alpha;\varphi_3^\alpha]+\eta_\alpha H_3[\varphi_1^\alpha;\varphi_3^\alpha]=0 \quad \alpha=m,i \text{ in } \Omega_\alpha, \tag{9}$$

$$\varphi_3^i(r)=\varphi_3^m(r)\big|_{r=a}, \tag{10}$$

$$\sigma_i J_3[\varphi_3^i]+\chi_i K_3[\varphi_1^i;\varphi_3^i]+\eta_i L_3[\varphi_1^i;\varphi_3^i]=\sigma_m J_3[\varphi_3^m]+\chi_m K_3[\varphi_1^m;\varphi_3^m]+\eta_m L_3[\varphi_1^m;\varphi_3^m]\Big|_{r=a}, \tag{11}$$

这里

$$F_1[\varphi_1]=-(r^2\varphi_{1rr}+r\varphi_{1r}-\varphi_1)/r^2,\quad F_3[\varphi_3]=-(r^2\varphi_{3rr}+r\varphi_{3r}-9\varphi_3)/r^2,$$

$$\begin{aligned}G_1[\varphi_1;\varphi_3]=&(-9r^4\varphi_{1r}^2\varphi_{1rr}-6r^4\varphi_{1rr}\varphi_{3r}^2-12r^4\varphi_{1r}\varphi_{3r}\varphi_{3rr}-r^2\varphi_1\varphi_{1r}^2-r^2\varphi_1^2\varphi_{1rr}+r\varphi_1^2\varphi_{1r}+18r\varphi_{1r}\varphi_3^2-18r^2\varphi_{1rr}\varphi_3^2\\&-36r^2\varphi_{1r}\varphi_3\varphi_{3r}-6r^4\varphi_{1r}\varphi_{1rr}\varphi_{3r}-3r^2\varphi_{1r}^2\varphi_{3rr}+r^2\varphi_{1r}^2\varphi_3-2r^2\varphi_1\varphi_3\varphi_{1rr}-2r\varphi_1\varphi_3\varphi_{1r}-r\varphi_1^2\varphi_{3r}\\&+r^2\varphi_1^2\varphi_{3rr}-3r^3\varphi_{1r}^3-6r^3\varphi_{1r}\varphi_{3r}^2-3r^3\varphi_{1r}^2\varphi_{3r}+2r^2\varphi_1\varphi_{3r}^2+3\varphi_1^3+54\varphi_1\varphi_3^2-2r^2\varphi_1\varphi_{1r}\varphi_{3r}-9\varphi_1^2\varphi_3)/4r^4,\end{aligned}$$

$$\begin{aligned}G_3[\varphi_1;\varphi_3]=&(-3r^4\varphi_{1r}^2\varphi_{1rr}-12r^4\varphi_{1r}\varphi_{1rr}\varphi_{3r}-6r^4\varphi_{1r}^2\varphi_{3rr}-r\varphi_1^2\varphi_{1r}+5r^2\varphi_1\varphi_{1r}^2+r^2\varphi_1^2\varphi_{1rr}\\&-9r^4\varphi_{3r}^2\varphi_{3rr}+2r\varphi_1^2\varphi_{3r}-2r^2\varphi_1^2\varphi_{3rr}-4r^2\varphi_1\varphi_{1r}\varphi_{3r}+9r\varphi_3^2\varphi_{3r}-9r^2\varphi_3\varphi_{3r}^2-9r^2\varphi_3^2\varphi_{3rr}\\&-r^3\varphi_{1r}^3-6r^3\varphi_{1r}^2\varphi_{3r}-3r^3\varphi_{3r}^3-3\varphi_1^3+54\varphi_1^2\varphi_3+18r^2\varphi_{1r}^2\varphi_3+243\varphi_3^3)/4r^4,\end{aligned}$$

$$J_1[\varphi_1]=\varphi_{1r},\quad J_3[\varphi_3]=\varphi_{3r},$$

$$K_1[\varphi_1;\varphi_3]=(3r^2\varphi_{1r}^3+6r^2\varphi_{1r}\varphi_{3r}^2+\varphi_1^2\varphi_{1r}+18\varphi_3^2\varphi_{1r}+3r^2\varphi_{1r}^2\varphi_{3r}+6\varphi_1\varphi_{1r}\varphi_3-\varphi_1^2\varphi_{3r})/4r^2,$$

$$K_3[\varphi_1;\varphi_3]=(r^2\varphi_{1r}^3+6r^2\varphi_{1r}^2\varphi_{3r}-\varphi_1^2\varphi_{1r}+3r^2\varphi_{3r}^3+2\varphi_1^2\varphi_{3r}+9\varphi_3^2\varphi_{3r})/4r^2,$$

$$\begin{aligned}L_1[\varphi_1;\varphi_3]=&(10r^4\varphi_{1r}^5+30r^4\varphi_{1r}\varphi_{3r}^4+2\varphi_1^4\varphi_{1r}+486\varphi_{1r}\varphi_3^4+60r^4\varphi_{1r}^3\varphi_{3r}^2+4r^2\varphi_1^2\varphi_{1r}^3+108r^2\varphi_{1r}^3\varphi_3^2+12r^2\varphi_1^2\varphi_{1r}\varphi_{3r}^2\\&+108r^2\varphi_{1r}\varphi_3^2\varphi_{3r}^2+44\varphi_1^2\varphi_{1r}\varphi_3^2+25r^4\varphi_{1r}^4\varphi_{3r}+36r^2\varphi_1\varphi_{1r}^3\varphi_3-6r^2\varphi_1^2\varphi_{1r}^2\varphi_{3r}+28\varphi_1^3\varphi_{1r}\varphi_3+30r^4\varphi_{1r}^2\varphi_{3r}^3\\&+36r^2\varphi_1\varphi_{1r}\varphi_3\varphi_{3r}^2+54r^2\varphi_{1r}^2\varphi_3^2\varphi_{3r}+324\varphi_1\varphi_{1r}\varphi_3^3-6r^2\varphi_1^2\varphi_{3r}^3-3\varphi_1^4\varphi_{3r}-54\varphi_1^2\varphi_3^2\varphi_{3r})/16r^4,\end{aligned}$$

$$\begin{aligned}L_3[\varphi_1;\varphi_3]=&(5r^4\varphi_{1r}^5+30r^4\varphi_{1r}^4\varphi_{3r}+30r^4\varphi_{1r}^3\varphi_{3r}^2+60r^4\varphi_{1r}^2\varphi_{3r}^3-18r^2\varphi_1^2\varphi_{1r}\varphi_{3r}^2+12r^2\varphi_1^2\varphi_{1r}^2\varphi_{3r}-3\varphi_1^4\varphi_{1r}\\&+18r^2\varphi_{1r}^3\varphi_3^2+108r^2\varphi_{1r}^2\varphi_3^2\varphi_{3r}-54\varphi_1^2\varphi_{1r}\varphi_3^2-2r^2\varphi_1^2\varphi_{1r}^3+36r^2\varphi_1\varphi_{1r}^2\varphi_3\varphi_{3r}+10r^4\varphi_{3r}^5+6\varphi_1^4\varphi_{3r}\\&+162\varphi_3^4\varphi_{3r}+12r^2\varphi_1^2\varphi_{3r}^3+36r^2\varphi_3^2\varphi_{3r}^3+44\varphi_1^2\varphi_3^2\varphi_{3r}+4\varphi_1^3\varphi_3\varphi_{3r})/16r^4,\end{aligned}$$

其中, 为简单起见, 将 $\varphi_1^\alpha,\varphi_3^\alpha$ 中的上标 α 省略, 记为 φ_1,φ_3; $\varphi_{1r},\varphi_{3r},\varphi_{1rr},\varphi_{3rr}$ 分别为 φ_1,φ_3 关于 r 的一阶及二阶导数; $H_1[\varphi_1;\varphi_3]$ 和 $H_3[\varphi_1;\varphi_3]$ 是关于 φ_1,φ_3 及它们导数的数学表达式 (详见附录 A).

4 电势分布的同伦分析解

同伦分析方法 [38] 的优点是不依赖于小参数, 克服了微扰展开法只能用于弱非线性问题的局限性. 首先, 引入同伦参数 p ($0\leqslant p\leqslant 1$), 并引入两个同伦函数 $\Phi_m(r,p)$ 和 $\Phi_i(r,p)$, 代入模函数方程 (6)—(11), 从而得到如下的同伦方程:

$$\sigma_\alpha F_1[\Phi_{1\alpha}(r,p)]+p\{\chi_\alpha G_1[\Phi_{1\alpha}(r,p);\Phi_{3\alpha}(r,p)]+\eta_\alpha H_1[\Phi_{1\alpha}(r,p);\Phi_{3\alpha}(r,p)]\}=0, \tag{12}$$

$$\sigma_\alpha F_3[\Phi_{3\alpha}(r,p)]+p\{\chi_\alpha G_3[\Phi_{1\alpha}(r,p);\Phi_{3\alpha}(r,p)]+\eta_\alpha H_3[\Phi_{1\alpha}(r,p);\Phi_{3\alpha}(r,p)]\}=0, \tag{13}$$

$$\Phi_{ki}(r,p)=\Phi_{km}(r,p)|_{r=a},\quad k=1,3, \tag{14}$$

$$\begin{aligned}&\sigma_i J_1[\Phi_{1i}(r,p)]+p\{\chi_i K_1[\Phi_{1i}(r,p);\Phi_{3i}(r,p)]+\eta_i L_1[\Phi_{1i}(r,p);\Phi_{3i}(r,p)]\}\\=&\sigma_m J_1[\Phi_{1m}(r,p)]+p\{\chi_m K_1[\Phi_{1m}(r,p);\Phi_{3m}(r,p)]+\eta_m L_1[\Phi_{1m}(r,p);\Phi_{3m}(r,p)]\}\big|_{r=a},\end{aligned} \tag{15}$$

$$\begin{aligned}&\sigma_i J_3[\Phi_{1i}(r,p)]+p\{\chi_i K_3[\Phi_{1i}(r,p);\Phi_{3i}(r,p)]+\eta_i L_3[\Phi_{1i}(r,p);\Phi_{3i}(r,p)]\}\\=&\sigma_m J_3[\Phi_{1m}(r,p)]+p\{\chi_m K_3[\Phi_{1m}(r,p);\Phi_{3m}(r,p)]+\eta_m L_3[\Phi_{1m}(r,p);\Phi_{3m}(r,p)]\}\big|_{r=a}.\end{aligned} \tag{16}$$

当 $p=0$ 时, $\Phi_{1\alpha}(r,0)$ 是

$$\sigma_\alpha F_1[\Phi_{1\alpha}(r,0)]=0 \quad \alpha=i,m, \tag{17}$$

$$\Phi_{1i}(r,0)=\Phi_{1m}(r,0)|_{r=a}, \tag{18}$$

$$\sigma_i J_1[\Phi_{1i}(r,0)]=\sigma_m J_1[\Phi_{1m}(r,0)]|_{r=a} \tag{19}$$

方程的解, 该解为

$$\Phi_{1i0}=-cE_0 r, \tag{20}$$

$$\Phi_{1m0}=-E_0(r+br^{-1}), \tag{21}$$

其中

$$c=\frac{2\sigma_m}{\sigma},\quad b=\frac{\sigma_m-\sigma_i}{\sigma}a^2,$$

$$\sigma=\sigma_i+\sigma_m,\quad \Phi_{1\alpha0}=\Phi_{1\alpha}(r,0)\quad \alpha=i,m.$$

而当 $p=1$ 时, $\Phi_{1\alpha}(r,1)$ 恰好是常微分方程边值问题 (6)—(8) 式的解. 因而 p 从 0 逐渐增大到 1 时, $\Phi_{1\alpha}(r,p)$ 从初始猜测解 $\Phi_{1\alpha0}(r)$ 变化到精确解 $\Phi_{1\alpha}(r)$, 即常微分方程边值问题 (6)—(8) 式的解. 在拓扑学中, 这种连续变化称为形变. 假定 $\Phi_{1\alpha}(r,p)$ 可以用关于 p 的泰勒级数表示

$$\Phi_{1\alpha}(r,p)=\Phi_{1\alpha}(r,0)+\sum_{k=1}^{\infty}\frac{1}{k!}\left.\frac{\partial^k\Phi_{1\alpha}(r,p)}{\partial p^k}\right|_{p=0}p^k. \tag{22}$$

令

$$\Phi_{1\alpha0}^{[k]}=\left.\frac{\partial^k\Phi_{1\alpha}(r,p)}{\partial p^k}\right|_{p=0},$$

则 (22) 式变为

$$\Phi_{1\alpha}(r,p)=\Phi_{1\alpha}(r,0)+\sum_{k=1}^{\infty}\frac{\Phi_{1\alpha0}^{[k]}}{k!}p^k. \tag{23}$$

于是常微分方程边值问题 (6)—(8) 式的解为

$$\Phi_{1\alpha}(r,1)=\Phi_{1\alpha}(r,0)+\sum_{k=1}^{\infty}\frac{\Phi_{1\alpha0}^{[k]}}{k!}. \tag{24}$$

同理, 当 $p=0$ 时, $\Phi_{3\alpha}(r,0)$ 是如下常微分方程边值问题

$$\sigma_\alpha F_3[\Phi_{3\alpha}(r,0)]=0 \quad \alpha=i,m, \tag{25}$$

$$\Phi_{3i}(r,0)=\Phi_{3m}(r,0)|_{r=a}, \tag{26}$$

$$\sigma_i J_3[\Phi_{3i}(r,0)]=\sigma_m J_3[\Phi_{3m}(r,0)]|_{r=a} \tag{27}$$

的解, 该解为

$$\Phi_{3i0}=\Phi_{3m0}=0. \tag{28}$$

同伦方程 (12)—(16) 对 p 求导, 并令 $p=0$, 可知一阶形变导数 $\Phi_{1\alpha0}^{[1]}$ 和 $\Phi_{3\alpha0}^{[1]}$ 所满足的方程为

$$\sigma_\alpha F_1[\Phi_{1\alpha}^{[1]}(r,0)]+\chi_\alpha G_1[\Phi_{1\alpha}(r,0);\Phi_{3\alpha}(r,0)]$$
$$+\eta_\alpha H_1[\Phi_{1\alpha}(r,0);\Phi_{3\alpha}(r,0)]=0, \tag{29}$$

$$\sigma_\alpha F_3[\Phi_{1\alpha}^{[1]}(r,0)]+\chi_\alpha G_3[\Phi_{1\alpha}(r,0);\Phi_{3\alpha}(r,0)]$$
$$+\eta_\alpha H_3[\Phi_{1\alpha}(r,0);\Phi_{3\alpha}(r,0)]=0, \tag{30}$$

$$\Phi_{ki0}^{[1]}=\left.\Phi_{km0}^{[1]}\right|_{r=a}\quad k=1,3, \tag{31}$$

$$\sigma_i J_1[\Phi_{1i}^{[1]}(r,0)]+\chi_i K_1[\Phi_{1i}(r,0);\Phi_{3i}(r,0)]$$
$$+\eta_i L_1[\Phi_{1i}(r,0);\Phi_{3i}(r,0)]$$
$$=\sigma_m J_1[\Phi_{1m}^{[1]}(r,0)]+\chi_m K_1[\Phi_{1m}(r,0);\Phi_{3m}(r,0)]$$
$$+\eta_m L_1[\Phi_{1m}(r,0);\Phi_{3m}(r,0)]|_{r=a}, \tag{32}$$

$$\sigma_i J_3[\Phi_{1i}^{[1]}(r,0)]+\chi_i K_3[\Phi_{1i}(r,0);\Phi_{3i}(r,0)]$$
$$+\eta_i L_3[\Phi_{1i}(r,0);\Phi_{3i}(r,0)]$$
$$=\sigma_m J_3[\Phi_{1m}^{[1]}(r,0)]+\chi_m K_3[\Phi_{1m}(r,0);\Phi_{3m}(r,0)]$$
$$+\eta_m L_3[\Phi_{1m}(r,0);\Phi_{3m}(r,0)]|_{r=a}. \tag{33}$$

代入初始猜测解, 则 $\Phi_{1\alpha0}^{[1]}$ 满足的方程及边界条件为

$$r^2\Phi_{1i0rr}^{[1]}+r\Phi_{1i0r}^{[1]}-\Phi_{1i0}^{[1]}=0, \tag{34}$$

$$\sigma_m(\Phi_{1m0rr}^{[1]}+r^{-1}\Phi_{1m0r}^{[1]}-r^{-2}\Phi_{1m0}^{[1]})$$
$$+\chi_m 4b^2E_0^3(2r^{-5}-br^{-7})+\eta_m 8b^2E_0^5(3r^{-5}$$
$$-3br^{-7}+3b^2r^{-9}-b^3r^{-11})=0, \tag{35}$$

$$\Phi_{1i0}^{[1]}=\left.\Phi_{1m0}^{[1]}\right|_{r=a}, \tag{36}$$

$$\sigma_i\Phi_{1i0r}^{[1]}-\chi_i c^3E_0^3-\eta_i c^5E_0^5$$
$$=\sigma_m\Phi_{1m0r}^{[1]}+\chi_m E_0^3(-1+2br^{-2}-2b^2r^{-4}+b^3r^{-6})$$
$$+\eta_m E_0^5(-1+3br^{-2}-6b^2r^{-4}+6b^3r^{-6}$$
$$-3b^4r^{-8}+b^5r^{-10})|_{r=a}. \tag{37}$$

$\Phi_{3\alpha0}^{[1]}$ 满足的方程及边界条件为

$$r^2\Phi_{3i0rr}^{[1]}+r\Phi_{3i0r}^{[1]}-9\Phi_{3i0}^{[1]}=0, \tag{38}$$

$$\sigma_m(\Phi_{3m0rr}^{[1]}+r^{-1}\Phi_{3m0r}^{[1]}-9r^{-2}\Phi_{3m0}^{[1]})-\chi_m 4bE_0^3r^{-3}$$
$$-\eta_m 8bE_0^5(r^{-3}+3b^2r^{-7}-b^3r^{-9})=0, \tag{39}$$

$$\Phi_{3i0}^{[1]}=\left.\Phi_{3m0}^{[1]}\right|_{r=a}, \tag{40}$$

$$\sigma_i\Phi_{3i0r}^{[1]}=\sigma_m\Phi_{3m0r}^{[1]}+\chi_m E_0^3(br^{-2}-b^2r^{-4})$$

$$+\eta_m E_0^5(2br^{-2}-3b^2r^{-4}+3b^3r^{-6}-2b^4r^{-8})\big|_{r=a}. \tag{41}$$

由方程及边界条件 (34)—(41) 式可解得

$$\Phi_{1i0}^{[1]}=C_1 r, \tag{42}$$

$$\begin{aligned}\Phi_{1m0}^{[1]}=&C_2r^{-1}+\frac{\chi_m b^2E_0^3}{\sigma_m}\Big(\frac{1}{6}br^{-5}-r^{-3}\Big)\\&+\frac{\eta_m b^2E_0^5}{\sigma_m}\Big(\frac{1}{10}b^3r^{-9}-\frac{1}{2}b^2r^{-7}\\&+br^{-5}-3r^{-3}\Big),\end{aligned} \tag{43}$$

$$\Phi_{3i0}^{[1]}=C_3r^3, \tag{44}$$

$$\begin{aligned}\Phi_{3m0}^{[1]}=&C_4r^{-3}-\frac{\chi_mE_0^3}{2\sigma_m}br^{-1}+\frac{\eta_mE_0^5}{\sigma_m}\Big(-br^{-1}\\&+\frac{3}{2}b^3r^{-5}-\frac{1}{5}b^4r^{-7}\Big),\end{aligned} \tag{45}$$

其中

$$\begin{aligned}C_1=&\frac{\chi_mE_0^3}{\sigma}\Big(-1+2ba^{-2}+\frac{1}{3}b^3a^{-6}+\frac{\chi_i}{\chi_m}c^3\Big)\\&-\frac{\eta_mE_0^5}{\sigma}(1-3ba^{-2}-2b^3a^{-6}-\frac{1}{5}b^5a^{-10}-\frac{\eta_i}{\eta_m}c^5),\\C_2=&\frac{\chi_mE_0^3}{\sigma}\Big(-a^2+2b+\frac{1}{3}b^3a^{-4}+\frac{\chi_i}{\chi_m}a^2c^3\Big)\\&-\frac{\eta_mE_0^5}{\sigma}\Big(a^2-3b-2b^3a^{-4}-\frac{1}{5}b^5a^{-8}-\frac{\eta_i}{\eta_m}a^2c^5\Big)\\&+\frac{\chi_mb^2E_0^3}{\sigma_m}\Big(a^{-2}-\frac{1}{6}ba^{-4}\Big)\\&-\frac{\eta_mb^2E_0^5}{\sigma_m}\Big(\frac{1}{10}b^3a^{-8}-\frac{1}{2}b^2a^{-6}+ba^{-4}-3a^{-2}\Big),\\C_3=&-\frac{\chi_mb^2a^{-6}E_0^3}{3\sigma}-\frac{\eta_mb^2E_0^5}{\sigma}\Big(a^{-6}+\frac{2}{5}b^2a^{-10}\Big),\\C_4=&-\frac{\chi_mb^2E_0^3}{3\sigma}-\frac{\eta_mb^2E_0^5}{\sigma}\Big(1+\frac{2}{5}b^2a^{-4}\Big)\\&+\frac{\chi_mba^2E_0^3}{2\sigma_m}+\frac{\eta_mbE_0^5}{\sigma_m}\Big(a^2-\frac{3}{2}b^2a^{-2}+\frac{1}{5}b^3a^{-4}\Big).\end{aligned}$$

因而电势的一阶近似解为

$$\begin{cases}\Phi_m(r,\theta)\approx[\Phi_{1m0}(r)+\Phi_{1m0}^{[1]}(r)]\cos\theta+[\Phi_{3m0}(r)\\\qquad+\Phi_{3m0}^{[1]}(r)]\cos3\theta,\\\Phi_i(r,\theta)\approx[\Phi_{1i0}(r)+\Phi_{1i0}^{[1]}(r)]\cos\theta+[\Phi_{3i0}(r)\\\qquad+\Phi_{3i0}^{[1]}(r)]\cos3\theta.\end{cases} \tag{46}$$

运用同样的方法, 并借助数学软件, 可以得到 $\Phi_\alpha(r,\theta)$ 的更高阶近似, 其形式为

$$\begin{aligned}\Phi_\alpha(r,\theta)\approx&\Big[\Phi_{1\alpha0}(r)+\sum_{k=1}^N\frac{\Phi_{1\alpha0}^{[k]}(r)}{k!}\Big]\cos\theta\\&+[\Phi_{3\alpha0}(r)+\sum_{k=1}^N\frac{\Phi_{3\alpha0}^{[k]}(r)}{k!}]\cos3\theta\\&(\alpha=m,i).\end{aligned} \tag{47}$$

5 结 论

本文利用同伦分析方法研究了一类由圆柱形杂质随机嵌入基质所形成的高阶弱非线性复合介质在外加直流电场作用下的电势分布问题, 用同伦分析方法得到了电势分布的一阶渐近解析解, 并指出借助于数学软件, 还可以得到更高阶的近似解析解. 此外, 一些已有研究结果为本文的特例, 而本文所得结果为它们的推广. 例如, 若令 $\eta_\alpha=0$ $(\alpha=i,m)$, 则本文所得的电势分布结果与 Wei 和 Gu[37] 所得的结果完全相同.

附录 A

$$\begin{aligned}H_1[\varphi_1;\varphi_3]=\Big(&-10r^5\varphi_{1r}^5-50r^6\varphi_{1r}^4\varphi_{1rr}-30r^5\varphi_{1r}\varphi_{3r}^4-30r^6\varphi_{1rr}\varphi_{3r}^4-120r^6\varphi_{1r}\varphi_{3r}^3\varphi_{3rr}+6r\varphi_1^4\varphi_{1r}-4r^2\varphi_1^3\varphi_{1r}^2-2r^2\varphi_1^4\varphi_{1rr}+1458r\varphi_{1r}\varphi_3^4\\&-486r^2\varphi_{1rr}\varphi_3^4-1944r^2\varphi_{1r}\varphi_3^3\varphi_{3r}-60r^5\varphi_{1r}^3\varphi_{3r}^2-180r^6\varphi_{1r}^2\varphi_{1rr}\varphi_{3r}^2-120r^6\varphi_{1r}^3\varphi_{3r}\varphi_{3rr}+4r^3\varphi_1^2\varphi_{1r}^3-6r^4\varphi_1\varphi_{1r}^4-12r^4\varphi_1^2\varphi_{1r}^2\varphi_{1rr}\\&+108r^3\varphi_{1r}^3\varphi_3^2-324r^4\varphi_{1r}^2\varphi_{1rr}\varphi_3^2-216r^4\varphi_{1r}^3\varphi_3\varphi_{3r}+12r^3\varphi_1^2\varphi_{1r}\varphi_{3r}^2-12r^4\varphi_1\varphi_{1r}^2\varphi_{3r}^2-12r^4\varphi_1^2\varphi_{1rr}\varphi_{3r}^2-24r^4\varphi_1^2\varphi_{1r}\varphi_{3r}\varphi_{3rr}\\&+108r^3\varphi_{1r}\varphi_3^2\varphi_{3r}^2-108r^4\varphi_{1rr}\varphi_3^2\varphi_{3r}^2-216r^4\varphi_{1r}\varphi_3\varphi_{3r}^3-216r^4\varphi_{1r}\varphi_3^2\varphi_{3r}\varphi_{3rr}+132r\varphi_1^2\varphi_{1r}\varphi_3^2+20r^2\varphi_1\varphi_{1r}^2\varphi_3^2-44r^2\varphi_1^2\varphi_{1rr}\varphi_3^2\\&-88r^2\varphi_1^2\varphi_{1r}\varphi_3\varphi_{3r}-25r^5\varphi_{1r}^4\varphi_{3r}-100r^6\varphi_{1r}^3\varphi_{1rr}\varphi_{3r}-25r^6\varphi_{1r}^4\varphi_{3rr}+36r^3\varphi_1\varphi_{1r}^3\varphi_3-27r^4\varphi_{1r}^4\varphi_3-108r^4\varphi_1\varphi_{1r}^2\varphi_{1rr}\varphi_3\\&-28r^4\varphi_1\varphi_{1r}^3\varphi_{3r}-6r^3\varphi_1^2\varphi_{1r}^2\varphi_{3r}+12r^4\varphi_1^2\varphi_{1r}\varphi_{1rr}\varphi_3+6r^4\varphi_1^2\varphi_{1r}^2\varphi_{3rr}+84r\varphi_1^3\varphi_{1r}\varphi_3-66r^2\varphi_1^2\varphi_{1r}^2\varphi_3-28r^2\varphi_1^3\varphi_{1rr}\varphi_3\\&-28r^2\varphi_1^3\varphi_{1r}\varphi_{3r}-30r^5\varphi_{1r}^2\varphi_{3r}^3-60r^6\varphi_{1r}\varphi_{1rr}\varphi_{3r}^3-90r^6\varphi_{1r}^2\varphi_{3r}^2\varphi_{3rr}+36r^3\varphi_1\varphi_{1r}\varphi_3\varphi_{3r}^2-126r^4\varphi_{1r}^2\varphi_3\varphi_{3r}^2-36r^4\varphi_1\varphi_{1rr}\varphi_3\varphi_{3r}^2\\&-36r^4\varphi_1\varphi_{1r}\varphi_{3r}^3-72r^4\varphi_1\varphi_{1r}\varphi_3\varphi_{3r}\varphi_{3rr}+54r^3\varphi_{1r}^2\varphi_3^2\varphi_{3r}-108r^4\varphi_{1r}\varphi_{1rr}\varphi_3^2\varphi_{3r}-54r^4\varphi_{1r}^2\varphi_3^2\varphi_{3rr}+972r\varphi_1\varphi_{1r}\varphi_3^3\\&-162r^2\varphi_{1r}^2\varphi_3^3-324r^2\varphi_1\varphi_{1rr}\varphi_3^3-972r^2\varphi_1\varphi_{1r}\varphi_3^2\varphi_{3r}-6r^3\varphi_1^2\varphi_{3r}^3+18r^4\varphi_1^2\varphi_{3r}^2\varphi_{3rr}-9r\varphi_1^4\varphi_{3r}+3r^2\varphi_1^4\varphi_{3rr}\\&-162r\varphi_1^2\varphi_3^2\varphi_{3r}+90r^2\varphi_1^2\varphi_3\varphi_{3r}^2+54r^2\varphi_1^2\varphi_3^2\varphi_{3rr}+6r^4\varphi_1\varphi_{3r}^4+10\varphi_1^5+2430\varphi_1\varphi_3^4+12r^2\varphi_1^3\varphi_{3r}^2+108r^2\varphi_1\varphi_3^2\varphi_{3r}^2\end{aligned}$$

$$
\begin{aligned}
&+476\varphi_1^3\varphi_3^2-59\varphi_1^4\varphi_3-810\varphi_1^2\varphi_3^3\Big)/16r^6,\\
H_3[\varphi_1;\varphi_3]=\Big(&-5r^5\varphi_{1r}^5-25r^6\varphi_{1r}^4\varphi_{1rr}-30r^5\varphi_{1r}^4\varphi_{3r}-120r^6\varphi_{1r}^3\varphi_{1rr}\varphi_{3r}-30r^6\varphi_{1r}^4\varphi_{3rr}-30r^5\varphi_{1r}^3\varphi_{3r}^2-90r^6\varphi_{1r}^2\varphi_{1rr}\varphi_{3r}^2-60r^6\varphi_{1r}^3\varphi_{3r}\varphi_{3rr}\\
&-60r^5\varphi_{1r}^2\varphi_{3r}^3-120r^6\varphi_{1r}\varphi_{1rr}\varphi_{3r}^3-180r^6\varphi_{1r}^2\varphi_{3r}^2\varphi_{3rr}-18r^3\varphi_1^2\varphi_{1r}\varphi_{3r}^2+18r^4\varphi_1^2\varphi_{1rr}\varphi_{3r}^2+36r^4\varphi_1^2\varphi_{1r}\varphi_{3r}\varphi_{3rr}+12r^3\varphi_1^2\varphi_{1r}^2\varphi_{3r}\\
&-24r^4\varphi_1\varphi_{1r}^3\varphi_{3r}-24r^4\varphi_1^2\varphi_{1r}\varphi_{1rr}\varphi_{3r}-12r^4\varphi_1^2\varphi_{1r}^2\varphi_{3rr}-9r\varphi_1^4\varphi_{1r}+18r^2\varphi_1^3\varphi_{1r}^2+3r^2\varphi_1^4\varphi_{1rr}+18r^3\varphi_{1r}^3\varphi_3^2-54r^4\varphi_{1r}^2\varphi_{1rr}\varphi_3^2\\
&-36r^4\varphi_{1r}^3\varphi_3\varphi_{3r}+108r^3\varphi_{1r}^2\varphi_3^2\varphi_{3r}-216r^4\varphi_{1r}\varphi_{1rr}\varphi_3^2\varphi_{3r}-108r^4\varphi_{1r}^2\varphi_3\varphi_{3r}^2-108r^4\varphi_{1r}^2\varphi_3^2\varphi_{3rr}-162r\varphi_1^2\varphi_{1r}\varphi_3^2+594r^2\varphi_1\varphi_{1r}^2\varphi_3^2\\
&+54r^2\varphi_1^2\varphi_{1rr}\varphi_3^2-12r^2\varphi_1^2\varphi_{1r}\varphi_3\varphi_{3r}-2r^3\varphi_1^2\varphi_{1r}^3+13r^4\varphi_1\varphi_{1r}^4+6r^4\varphi_1^2\varphi_{1r}^2\varphi_{1rr}+36r^3\varphi_1\varphi_{1r}^2\varphi_3\varphi_{3r}-27r^4\varphi_1\varphi_{1r}\varphi_{1rr}\varphi_3\varphi_{3r}\\
&-36r^4\varphi_1\varphi_{1r}^2\varphi_3\varphi_{3rr}-10r^5\varphi_{3r}^5-50r^6\varphi_{3r}^4\varphi_{3rr}+18r\varphi_1^4\varphi_{3r}-24r^2\varphi_1^3\varphi_{1r}\varphi_{3r}-6r^2\varphi_1^4\varphi_{3rr}+486r\varphi_3^4\varphi_{3r}-108r^4\varphi_3^2\varphi_{3r}^2\varphi_{3rr}\\
&-324r^2\varphi_3^3\varphi_{3r}^2-162r^2\varphi_3^4\varphi_{3rr}+12r^3\varphi_1^2\varphi_{3r}^3-24r^4\varphi_1\varphi_{1r}\varphi_{3r}^3-36r^4\varphi_1^2\varphi_{3r}^2\varphi_{3rr}+36r^3\varphi_3^2\varphi_{3r}^3-50r^4\varphi_3\varphi_{3r}^4+132r\varphi_1^2\varphi_3^2\varphi_{3r}\\
&+20r^2\varphi_1^2\varphi_3\varphi_{3r}^2-88r^2\varphi_1\varphi_{1r}\varphi_3^2\varphi_{3r}-44r^2\varphi_1^2\varphi_3^2\varphi_{3rr}+12r\varphi_1^3\varphi_3\varphi_{3r}-10r^2\varphi_1^3\varphi_{3r}^2-4r^2\varphi_1^3\varphi_3\varphi_{3rr}+108r^2\varphi_1^2\varphi_{1r}^2\varphi_3\\
&+18r^4\varphi_1\varphi_{1r}^2\varphi_{3r}^2-15\varphi_1^5+270\varphi_1^4\varphi_3-666\varphi_1^3\varphi_3^2+4284\varphi_1^2\varphi_3^3+54r^4\varphi_{1r}^4\varphi_3+7290\varphi_3^5+972r^2\varphi_{1r}^2\varphi_3^3\Big)/16r^6.
\end{aligned}
$$

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Electrostatic potential distribution of high-order weakly nonlinear composites under external direct current electric field*

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Abstract

By using the homotopy analysis method (HAM), the electrostatic potential distribution problems of a type of high-order weakly nonlinear composite with a cylindrical inclusion randomly embedded in a host medium, which obeyes a current-field constitutive relation of $\boldsymbol{J} = \sigma \boldsymbol{E} + \chi \left|\boldsymbol{E}\right|^2 \boldsymbol{E} + \eta \left|\boldsymbol{E}\right|^4 \boldsymbol{E}$, are investigated under the action of an external direct current electric field. With the mode expansion method, the current-field constitutive relation and their boundary conditions are transformed into a series of boundary value problems of nonlinear ordinary differential equations. Then the HAM is used to solve the boundary value problems of nonlinear ordinary differential equations and the asymptotic analytical solutions of electrostatic potential distribution in the inclusion and the host regions are given.

Keywords: high-order weakly nonlinear composites, mode function expansion method, homotopy analysis method, electrostatic potential distribution

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