两自由度弱非线性耦合系统的一阶近似 Lie 对称性与近似守恒量*

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采用变劲度系数的耦合弹簧构建一实际的两自由度弱非线性耦合系统, 用近似 Lie 对称性理论研究系统的一阶近似 Lie 对称性与近似守恒量, 得到 6 个一阶近似 Lie 对称性和一阶近似守恒量, 其中 1 个一阶近似守恒量实为系统的精确守恒量, 4 个一阶近似守恒量为平凡的一阶近似守恒量, 只有 1 个一阶近似守恒量为稳定的一阶近似守恒量.

关键词: 两自由度弱非线性耦合系统, 近似 Lie 对称性, 近似守恒量

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1 引 言

力学系统的对称性与守恒量之间存在着密切的联系, 关于力学系统对称性与守恒量的研究已渗透到现代数学、力学、物理学等各个领域. 寻求力学系统的对称性和守恒量已成为近代分析力学的一大热点问题 [1−8]. 但事实上, 许多力学系统会受到各种各样的微扰, 力学系统的某些参数会随着位移、速度和时间发生微小的变化, 通过实际力学模型建立的运动微分方程总是近似的、非线性的, 因此研究受非线性微扰的耦合系统的近似对称性和近似守恒量对于研究力学系统的特性以及得到方程的近似解至关重要. 近年来关于常微分方程、偏微分方程近似对称性和近似守恒量的研究已取得不少的成果 [9−25]. 目前研究近似对称性和近似守恒量主要采用近似 Lie 对称性理论 [9] 和近似 Noether 对称性理论 [10]. 引进近似的群无限小变换, 微分方程在此变换下近似保持不变则为近似 Lie 对称性; 哈密顿作用量在此变换下近似保持不变则为近似 Noether 对称性, 所得的守恒量为近似守恒量. 本文首先采用变劲度系数的耦合弹簧构建一实际的两自由度弱非线性耦合系统, 用近似 Lie 对称性理论研究该系统的一阶近似 Lie 对称性与近似守恒量, 得到了 6 个一阶近似 Lie 对称性和一阶近似守恒量, 其中 1 个一阶近似守恒量实为系统的精确守恒量, 4 个一阶近似守恒量为平凡的一阶近似守恒量, 只有 1 个一阶近似守恒量为稳定的一阶近似守恒量. 文中构造的力学系统典型而实际, 广泛存在于力学、振动学、原子与分子物理等各个领域.

2 两自由度弱非线性耦合系统的构造与其运动微分方程

设力学系统由两个质量为 m 的质点和三根劲度系数分别为 k_1, k_2, k_3 的弹簧组成 (如图 1), 边上两弹簧的劲度系数相同 ($k_1 = k_3 = k$), 中间耦合弹簧的劲度系数 k_2 与其伸缩量 $(x_2 - x_1)$ 存在弱线性关系, 即

$$k_2 = \frac{3}{2}k[1+\varepsilon(x_2-x_1)], \tag{1}$$

其中 k, ε 为常数, 且 $0 < \varepsilon \ll 1$, x_1, x_2 分别表示两质点相对其平衡位置的位移, 忽略阻力, 则系统的势

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能为

$$V = \frac{1}{2}kx_1^2 + \frac{3}{4}k[1+\varepsilon(x_2-x_1)](x_2-x_1)^2 + \frac{1}{2}kx_2^2, \tag{2}$$

Lagrange 函数可表示为

$$L = \frac{m}{2}(\dot{x}_1^2+\dot{x}_2^2) - \frac{1}{4}k(5x_1^2+5x_2^2-6x_1x_2) + \varepsilon\frac{3}{4}k(x_1-x_2)^3, \tag{3}$$

系统的运动微分方程为

$$\begin{aligned}\ddot{x}_1 &= -\frac{5}{2}\omega_0^2x_1 + \frac{3}{2}\omega_0^2x_2 + \varepsilon\frac{9}{4}\omega_0^2(x_1-x_2)^2 \\ &= g_1 = g_1(\varepsilon^0) + \varepsilon g_1(\varepsilon^1),\end{aligned} \tag{4a}$$

$$\begin{aligned}\ddot{x}_2 &= -\frac{5}{2}\omega_0^2x_2 + \frac{3}{2}\omega_0^2x_1 - \varepsilon\frac{9}{4}\omega_0^2(x_1-x_2)^2 \\ &= g_2 = g_2(\varepsilon^0) + \varepsilon\, g_2(\varepsilon^1),\end{aligned} \tag{4b}$$

其中 $\omega_0^2 = \dfrac{k}{m}$ 为常数, g_1, g_2 为广义加速度, $g_1(\varepsilon^0)$, $g_1(\varepsilon^1)$, $g_2(\varepsilon^0)$, $g_2(\varepsilon^1)$ 分别表示 g_1, g_2 中 ε^0, ε^1 项的系数 (下文表示类同), 且

$$\begin{aligned}g_1(\varepsilon^0) &= \omega_0^2(-\frac{5}{2}x_1+\frac{3}{2}x_2), \\ g_1(\varepsilon^1) &= \frac{9}{4}\omega_0^2(x_1-x_2)^2, \\ g_2(\varepsilon^0) &= \omega_0^2(-\frac{5}{2}x_2+\frac{3}{2}x_1), \\ g_2(\varepsilon^1) &= -\frac{9}{4}\omega_0^2(x_1-x_2)^2,\end{aligned} \tag{5}$$

ε 可称为非线性耦合系数, 因 $0<\varepsilon\ll 1$, 故系统 (4) 属两自由度弱非线性耦合系统.

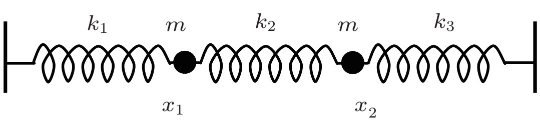

图 1　两自由度弱非线性耦合系统

3　系统的一阶近似 Lie 对称性

引进近似的群无限小变换

$$\begin{aligned}t^* &= t + \delta\tau(t,x_s,\dot{x}_s,\varepsilon), \\ x_s^*(t^*) &= x_s(t) + \delta\xi_s(t,x_s,\dot{x}_s,\varepsilon) \quad (s=1,2),\end{aligned} \tag{6}$$

其中 δ 为无限小参数, τ, ξ_s 为无限小变换生成元. (6) 式的无限小生成元向量为

$$\boldsymbol{X}^{(0)} = \tau\frac{\partial}{\partial t} + \sum_{s=1}^{2}\xi_s\frac{\partial}{\partial x_s}, \tag{7}$$

(7) 式的一次扩展为

$$\boldsymbol{X}^{(1)} = \boldsymbol{X}^{(0)} + \sum_{s=1}^{2}(\dot{\xi}_s - \dot{x}_s\dot{\tau})\frac{\partial}{\partial \dot{x}_s}, \tag{8}$$

二次扩展为

$$\boldsymbol{X}^{(2)} = \boldsymbol{X}^{(1)} + \sum_{s=1}^{2}(\ddot{\xi}_s - \dot{x}_s\ddot{\tau} - 2\ddot{x}_s\dot{\tau})\frac{\partial}{\partial \ddot{x}_s}. \tag{9}$$

(6)—(9) 式中

$$\tau = \tau_0 + \varepsilon\tau_1, \xi_s = \xi_{s0} + \varepsilon\xi_{s1} \quad (s=1,2). \tag{10}$$

并有

$$\dot{\tau} = \dot{\tau}_0 + \varepsilon\dot{\tau}_1, \ddot{\tau} = \ddot{\tau}_0 + \varepsilon\ddot{\tau}_1, \tag{11a}$$

$$\dot{\xi}_s = \dot{\xi}_{s0} + \varepsilon\dot{\xi}_{s1}, \ddot{\xi}_s = \ddot{\xi}_{s0} + \varepsilon\ddot{\xi}_{s1} \quad (s=1,2). \tag{11b}$$

对于自治系统, 可设 τ, ξ_s 不显含时间 t, 则

$$\begin{aligned}\dot{\tau}_i &= \tau_{ix_1}\dot{x}_1 + \tau_{ix_2}\dot{x}_2 + g_1\tau_{i\dot{x}_1} + g_2\tau_{i\dot{x}_2} \\ &= \dot{\tau}_i(\varepsilon^0) + \varepsilon\dot{\tau}_i(\varepsilon^1) \quad (i=0,1),\end{aligned} \tag{12a}$$

$$\begin{aligned}\dot{\xi}_{si} &= \xi_{six_1}\dot{x}_1 + \xi_{six_2}\dot{x}_2 + g_1\xi_{si\dot{x}_1} + g_2\xi_{si\dot{x}_2} \\ &= \dot{\xi}_{si}(\varepsilon^0) + \varepsilon\dot{\xi}_{si}(\varepsilon^1) \quad (s=1,2; i=0,1),\end{aligned} \tag{12b}$$

$$\begin{aligned}\ddot{\tau}_i &= \dot{x}_1^2\tau_{ix_1x_1} + \dot{x}_2\dot{x}_1\tau_{ix_1x_2} + g_1\dot{x}_1\tau_{ix_1\dot{x}_1} + g_2\dot{x}_1\tau_{ix_1\dot{x}_2} \\ &\quad + g_1\tau_{ix_1} + \dot{x}_1\dot{x}_2\tau_{ix_2x_1} + \dot{x}_2^2\tau_{ix_2x_2} + g_1\dot{x}_2\tau_{ix_2\dot{x}_1} \\ &\quad + g_2\dot{x}_2\tau_{ix_2\dot{x}_2} + g_2\tau_{ix_2} + g_1\dot{x}_1\tau_{i\dot{x}_1x_1} + g_1\dot{x}_2\tau_{i\dot{x}_1x_2} \\ &\quad + g_1^2\tau_{i\dot{x}_1\dot{x}_1} + g_1g_2\tau_{i\dot{x}_1\dot{x}_2} + \dot{g}_1\tau_{i\dot{x}_1} + g_2\dot{x}_1\tau_{i\dot{x}_2x_1} \\ &\quad + g_2\dot{x}_2\tau_{i\dot{x}_2x_2} + g_1g_2\tau_{i\dot{x}_2\dot{x}_1} + g_2^2\tau_{i\dot{x}_2\dot{x}_2} + \dot{g}_2\tau_{i\dot{x}_2} \\ &= \ddot{\tau}_i(\varepsilon^0) + \varepsilon\ddot{\tau}_i(\varepsilon^1) + \varepsilon^2\ddot{\tau}_i(\varepsilon^2) \quad (i=0,1),\end{aligned} \tag{12c}$$

$$\begin{aligned}\ddot{\xi}_{si} &= \dot{x}_1^2\xi_{six_1x_1} + \dot{x}_2\dot{x}_1\xi_{six_1x_2} + g_1\dot{x}_1\xi_{six_1\dot{x}_1} + g_2\dot{x}_1\xi_{six_1\dot{x}_2} \\ &\quad + g_1\xi_{six_1} + \dot{x}_1\dot{x}_2\xi_{six_2x_1} + \dot{x}_2^2\xi_{six_2x_2} + g_1\dot{x}_2\xi_{six_2\dot{x}_1} \\ &\quad + g_2\dot{x}_2\xi_{six_2\dot{x}_2} + g_2\xi_{six_2} + g_1\dot{x}_1\xi_{si\dot{x}_1x_1} + g_1\dot{x}_2\xi_{si\dot{x}_1x_2} \\ &\quad + g_1^2\xi_{si\dot{x}_1\dot{x}_1} + g_1g_2\xi_{si\dot{x}_1\dot{x}_2} + \dot{g}_1\xi_{si\dot{x}_1} + g_2\dot{x}_1\xi_{si\dot{x}_2x_1} \\ &\quad + g_2\dot{x}_2\xi_{si\dot{x}_2x_2} + g_1g_2\xi_{si\dot{x}_2\dot{x}_1} + g_2^2\xi_{si\dot{x}_2\dot{x}_2} + \dot{g}_2\xi_{si\dot{x}_2} \\ &= \ddot{\xi}_{si}(\varepsilon^0) + \varepsilon\ddot{\xi}_{si}(\varepsilon^1) \\ &\quad + \varepsilon^2\ddot{\xi}_{si}(\varepsilon^2) \quad (s=1,2; i=0,1),\end{aligned} \tag{12d}$$

(12a) 式中的 τ_{ix_1} 表示 τ_i 对 x_1 的偏导, 其他表示类同.

根据近似 Lie 对称性理论 [9], 运动微分方程 (4) 的一阶近似 Lie 对称性是指 (4) 式在近似的群无限小变换 (6) 式下近似保持不变, 即

$$\boldsymbol{X}^{(2)}(\ddot{x}_1 - g_1) = O(\varepsilon^2), \tag{13a}$$

$$\boldsymbol{X}^{(2)}(\ddot{x}_2 - g_2) = O(\varepsilon^2). \tag{13b}$$

利用 (7)—(9) 式, (13a) 和 (13b) 式可表示成

$$\begin{aligned}&\ddot{\xi}_1-\dot{x}_1\ddot{\tau}-2g_1\dot{\tau}+\frac{5}{2}\omega_0^2\xi_1-\frac{3}{2}\omega_0^2\xi_2\\&\quad-\frac{9}{2}\varepsilon\omega_0^2(x_1-x_2)\xi_1+\frac{9}{2}\varepsilon\omega_0^2(x_1-x_2)\xi_2\\&=O(\varepsilon^2),\end{aligned}\tag{14a}$$

$$\begin{aligned}&\ddot{\xi}_2-\dot{x}_2\ddot{\tau}-2g_2\dot{\tau}+\frac{5}{2}\omega_0^2\xi_2-\frac{3}{2}\omega_0^2\xi_1\\&\quad+\frac{9}{2}\varepsilon\omega_0^2(x_1-x_2)\xi_1-\frac{9}{2}\varepsilon\omega_0^2(x_1-x_2)\xi_2\\&=O(\varepsilon^2).\end{aligned}\tag{14b}$$

将 (4), (12) 式代入 (14) 式并展开, 令 ε^0, ε^1 项的系数为 0, 可得到 4 个关于 τ_0, τ_1, ξ_{10}, ξ_{11}, ξ_{20}, ξ_{21} 的二阶微分方程组.

令 (14a) 式中 ε^0 项的系数为 0, 可得

$$\begin{aligned}&\ddot{\xi}_{10}(\varepsilon^0)-\dot{x}_1\ddot{\tau}_0(\varepsilon^0)-2g_1(\varepsilon^0)\dot{\tau}_0(\varepsilon^0)\\&\quad+\frac{5}{2}\omega_0^2\xi_{10}-\frac{3}{2}\omega_0^2\xi_{20}=0;\end{aligned}\tag{15}$$

令 (14a) 式中 ε^1 项系数为 0, 可得

$$\begin{aligned}&\ddot{\xi}_{10}(\varepsilon^1)+\ddot{\xi}_{11}(\varepsilon^0)-\dot{x}_1\ddot{\tau}_0(\varepsilon^1)-\dot{x}_1\ddot{\tau}_1(\varepsilon^0)\\&-2g_1(\varepsilon^0)\dot{\tau}_0(\varepsilon^1)-2g_1(\varepsilon^0)\dot{\tau}_1(\varepsilon^0)-2g_1(\varepsilon^1)\dot{\tau}_0(\varepsilon^0)\\&+\frac{5}{2}\omega_0^2\xi_{11}-\frac{3}{2}\omega_0^2\xi_{21}-\frac{9}{2}\omega_0^2(x_1-x_2)\xi_{10}\\&+\frac{9}{2}\omega_0^2(x_1-x_2)\xi_{20}=0;\end{aligned}\tag{16}$$

令 (14b) 式中 ε^0 项的系数为 0, 可得

$$\begin{aligned}&\ddot{\xi}_{20}(\varepsilon^0)-\dot{x}_2\ddot{\tau}_0(\varepsilon^0)-2g_2(\varepsilon^0)\dot{\tau}_0(\varepsilon^0)\\&\quad+\frac{5}{2}\omega_0^2\xi_{20}-\frac{3}{2}\omega_0^2\xi_{10}=0;\end{aligned}\tag{17}$$

令 (14b) 式中 ε^1 项系数为 0, 可得

$$\begin{aligned}&\ddot{\xi}_{20}(\varepsilon^1)+\ddot{\xi}_{21}(\varepsilon^0)-\dot{x}_2\ddot{\tau}_0(\varepsilon^1)-\dot{x}_2\ddot{\tau}_1(\varepsilon^0)\\&-2g_2(\varepsilon^0)\dot{\tau}_0(\varepsilon^1)-2g_2(\varepsilon^0)\dot{\tau}_1(\varepsilon^0)-2g_2(\varepsilon^1)\dot{\tau}_0(\varepsilon^0)\\&+\frac{5}{2}\omega_0^2\xi_{21}-\frac{3}{2}\omega_0^2\xi_{11}+\frac{9}{2}\omega_0^2(x_1-x_2)\xi_{10}\\&-\frac{9}{2}\omega_0^2(x_1-x_2)\xi_{20}=0.\end{aligned}\tag{18}$$

将 (5) 式, (12) 式代入 (15)—(18) 式, 可解得如下 6 组 τ_0, τ_1, ξ_{10}, ξ_{11}, ξ_{20}, ξ_{21}.

$$\tau_0=-1,\xi_{10}=\xi_{20}=0,\tau_1=\xi_{11}=\xi_{21}=0;\tag{19a}$$

$$\tau_0=\xi_{10}=\xi_{20}=0,\tau_1=-1,\xi_{11}=\xi_{21}=0;\tag{19b}$$

$$\begin{aligned}&\tau_0=\xi_{10}=\xi_{20}=0,\tau_1=-\frac{1}{2},\\&\xi_{11}=\frac{1}{2}\dot{x}_2,\xi_{21}=\frac{1}{2}\dot{x}_1;\end{aligned}\tag{19c}$$

$$\begin{aligned}&\tau_0=\xi_{10}=\xi_{20}=0,\tau_1=-\frac{1}{2},\\&\xi_{11}=-\frac{1}{2}\dot{x}_2,\xi_{21}=-\frac{1}{2}\dot{x}_1;\end{aligned}\tag{19d}$$

$$\begin{aligned}&\tau_0=\xi_{10}=\xi_{20}=0,\tau_1=0,\\&\xi_{11}=2(x_1+x_2)\dot{x}_1-2(x_1-x_2)(\dot{x}_1+\dot{x}_2),\\&\xi_{21}=-2(x_1+x_2)\dot{x}_2-2(x_1-x_2)(\dot{x}_1+\dot{x}_2);\end{aligned}\tag{19e}$$

$$\begin{aligned}&\tau_0=0,\ \xi_{10}=2(x_1+x_2)\dot{x}_1-2(x_1-x_2)(\dot{x}_1+\dot{x}_2),\\&\xi_{20}=-2(x_1+x_2)\dot{x}_2-2(x_1-x_2)(\dot{x}_1+\dot{x}_2),\ \tau_1=0,\\&\xi_{11}=\frac{3}{16}[4(x_1-x_2)(x_1+x_2)(\dot{x}_1+\dot{x}_2)\\&\qquad+4(x_1-x_2)(x_1+x_2)(\dot{x}_1-\dot{x}_2)\\&\qquad+(\dot{x}_1-\dot{x}_2)^2(\dot{x}_1+\dot{x}_2)+(\dot{x}_1-\dot{x}_2)(\dot{x}_1+\dot{x}_2)^2\\&\qquad-3(x_1+x_2)^2(\dot{x}_1-\dot{x}_2)],\\&\xi_{21}=\frac{3}{16}[-4(x_1-x_2)(x_1+x_2)(\dot{x}_1+\dot{x}_2)\\&\qquad+4(x_1-x_2)(x_1+x_2)(\dot{x}_1-\dot{x}_2)\\&\qquad+(\dot{x}_1-\dot{x}_2)^2(\dot{x}_1+\dot{x}_2)-(\dot{x}_1-\dot{x}_2)(\dot{x}_1+\dot{x}_2)^2\\&\qquad+3(x_1+x_2)^2(\dot{x}_1-\dot{x}_2)].\end{aligned}\tag{19f}$$

若 $\tau_1=\xi_{11}=\xi_{21}=0$, 则相应的对称性为精确的 Lie 对称性, 所得的守恒量为精确守恒量; 若 $\tau_0=\xi_{10}=\xi_{20}=0$, 则相应的对称性为平凡的一阶近似 Lie 对称性, 所得的守恒量为平凡的一阶近似守恒量; 若 $\tau_0, \xi_{10}, \xi_{20}$ 不全为 0, 同时 $\tau_1, \xi_{11}, \xi_{21}$ 也不全为 0, 则相应的对称性为稳定的一阶近似 Lie 对称性, 所得的守恒量为稳定的一阶近似守恒量. 上述所得 6 个对称性中, (19a) 为精确的 Lie 对称性, (19b)—(19e) 为平凡的一阶近似 Lie 对称性, (19f) 为稳定的一阶近似 Lie 对称性.

4 系统的一阶近似守恒量

下面讨论系统的一阶近似守恒量. 若存在规范函数

$$G=G(x_s,\dot{x}_s,\varepsilon)=G_0+\varepsilon G_1\quad(s=1,2)\tag{20}$$

满足

$$\begin{aligned}&\frac{\partial L}{\partial t}\tau+\sum_{s=1}^{2}\frac{\partial L}{\partial x_s}\xi_s+\sum_{s=1}^{2}\frac{\partial L}{\partial \dot{x}_s}\dot{\xi}_s\\&+\left(L-\sum_{s=1}^{2}\frac{\partial L}{\partial \dot{x}_s}\dot{x}_s\right)\dot{\tau}=-\dot{G},\end{aligned}\tag{21}$$

则系统存在一阶近似守恒量 $I = I_0 + \varepsilon I_1$,

$$I = L\tau + \sum_{s=1}^{2} \frac{\partial L}{\partial \dot{x}_s}(\xi_s - \dot{x}_s \tau) + G \tag{22}$$

满足 $\frac{\mathrm{d}I}{\mathrm{d}t} = O(\varepsilon^2)$, 即

$$\begin{aligned} &\frac{\mathrm{d}I_0}{\mathrm{d}t}(\varepsilon^0) = 0, \\ &\frac{\mathrm{d}I_0}{\mathrm{d}t}(\varepsilon^1) + \frac{\mathrm{d}I_1}{\mathrm{d}t}(\varepsilon^0) = 0. \end{aligned} \tag{23}$$

对于自治系统, $\frac{\partial L}{\partial t} = 0$. 将 (3), (4), (12a), (12b) 式代入 (21) 式, 并比较等式两边 ε^0, ε^1 项的系数, 可得关于 G_0, G_1 的两个方程:

$$\begin{aligned} &mg_1(\varepsilon^0)\xi_{10} + mg_2(\varepsilon^0)\xi_{20} + m\dot{x}_1\dot{\xi}_{10}(\varepsilon^0) \\ &+ m\dot{x}_2\dot{\xi}_{20}(\varepsilon^0) - H(\varepsilon^0)\dot{\tau}_0(\varepsilon^0) = -\dot{G}_0(\varepsilon^0), \end{aligned} \tag{24a}$$

$$\begin{aligned} &mg_1(\varepsilon^1)\xi_{10} + mg_1(\varepsilon^0)\xi_{11} + mg_2(\varepsilon^1)\xi_{20} + mg_2(\varepsilon^0)\xi_{21} \\ &+ m\dot{x}_1\dot{\xi}_{10}(\varepsilon^1) + m\dot{x}_1\dot{\xi}_{11}(\varepsilon^0) + m\dot{x}_2\dot{\xi}_{20}(\varepsilon^1) + m\dot{x}_2\dot{\xi}_{21}(\varepsilon^0) \\ &- H(\varepsilon^1)\dot{\tau}_0(\varepsilon^0) - H(\varepsilon^0)\dot{\tau}_0(\varepsilon^1) - H(\varepsilon^0)\dot{\tau}_1(\varepsilon^0) \\ &= -\dot{G}_0(\varepsilon^1) - \dot{G}_1(\varepsilon^0), \end{aligned} \tag{24b}$$

其中 H 为系统的 Hamilton 函数

$$\begin{aligned} H = &\frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{4}k(5x_1^2 + 5x_2^2 - 6x_1x_2) \\ &- \varepsilon\frac{3}{4}k(x_1 - x_2)^3 \\ = &H(\varepsilon^0) + \varepsilon H(\varepsilon^1). \end{aligned} \tag{25}$$

将 (5), (19), (25) 式代入 (24) 式可解得与 (19) 式相应的 6 组规范函数 G_0, G_1:

$$G_0 = 0, G_1 = 0; \tag{26a}$$

$$G_0 = 0, G_1 = 0; \tag{26b}$$

$$\begin{aligned} &G_0 = 0, \\ &G_1 = -\frac{1}{2}m\dot{x}_1\dot{x}_2 - \frac{3k}{8}x_1^2 - \frac{3k}{8}x_2^2 + \frac{5k}{4}x_1x_2; \end{aligned} \tag{26c}$$

$$\begin{aligned} &G_0 = 0, \\ &G_1 = \frac{1}{2}m\dot{x}_1\dot{x}_2 + \frac{3k}{8}x_1^2 + \frac{3k}{8}x_2^2 - \frac{5k}{4}x_1x_2; \end{aligned} \tag{26d}$$

$$\begin{aligned} &G_0 = 0, \\ &G_1 = k(x_1 - x_2)(x_1 + x_2)^2 + m(x_1 - x_2)(\dot{x}_1 + \dot{x}_2)^2 \\ &\quad - m(x_1 + x_2)(\dot{x}_1^2 - \dot{x}_2^2); \end{aligned} \tag{26e}$$

$$\begin{aligned} &G_0 = k(x_1 - x_2)(x_1 + x_2)^2 + m(x_1 - x_2)(\dot{x}_1 + \dot{x}_2)^2 \\ &\quad - m(x_1 + x_2)(\dot{x}_1^2 - \dot{x}_2^2), \\ &G_1 = \frac{3}{32}[-8m(x_1^2 - x_2^2)(\dot{x}_1^2 - \dot{x}_2^2) \\ &\quad - 3m(\dot{x}_1 - \dot{x}_2)^2(\dot{x}_1 + \dot{x}_2)^2 \\ &\quad - 8k(x_1^2 - x_2^2)^2 \\ &\quad + 3m(x_1 + x_2)^2(\dot{x}_1 - \dot{x}_2)^2]. \end{aligned} \tag{26f}$$

将 (3), (19), (26) 式代入 (22) 式, 得 6 个一阶近似守恒量:

$$\begin{aligned} I^1 = &\frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{4}k(5x_1^2 + 5x_2^2 - 6x_1x_2) \\ &- \varepsilon\frac{3}{4}k(x_1 - x_2)^3; \end{aligned} \tag{27a}$$

$$I^2 = \varepsilon[\frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{4}k(5x_1^2 + 5x_2^2 - 6x_1x_2)]; \tag{27b}$$

$$I^3 = \varepsilon[\frac{m}{4}(\dot{x}_1 + \dot{x}_2)^2 + \frac{k}{4}(x_1 + x_2)^2]; \tag{27c}$$

$$I^4 = \varepsilon[\frac{m}{4}(\dot{x}_1 - \dot{x}_2)^2 + k(x_1 - x_2)^2]; \tag{27d}$$

$$\begin{aligned} I^5 = &\varepsilon[k(x_1 - x_2)(x_1 + x_2)^2 - m(x_1 - x_2)(\dot{x}_1 + \dot{x}_2)^2 \\ &+ m(x_1 + x_2)(\dot{x}_1^2 - \dot{x}_2^2)]; \end{aligned} \tag{27e}$$

$$\begin{aligned} I^6 = &k(x_1 - x_2)(x_1 + x_2)^2 - m(x_1 - x_2)(\dot{x}_1 + \dot{x}_2)^2 \\ &+ m(x_1 + x_2)(\dot{x}_1^2 - \dot{x}_2^2) \\ &+ \frac{3\varepsilon}{32}[8m(x_1^2 - x_2^2)(\dot{x}_1^2 - \dot{x}_2^2) \\ &+ m(\dot{x}_1 - \dot{x}_2)^2(\dot{x}_1 + \dot{x}_2)^2 \\ &- 8k(x_1^2 - x_2^2)^2 - 3m(x_1 + x_2)^2(\dot{x}_1 - \dot{x}_2)^2]. \end{aligned} \tag{27f}$$

(27) 式表示的 6 个守恒量可分为三类: I^1 为系统的精确守恒量, 相应的对称性也为精确 Lie 对称性, 它是系统的总能量; I^2, I^3, I^4, I^5 是平凡的一阶近似守恒量, 相应的对称性也为平凡的一阶近似 Lie 对称性; I^6 为系统稳定的一阶近似守恒量, 相应的对称性也为稳定的一阶近似 Lie 对称性.

5 结 论

本文首先采用变劲度系数的耦合弹簧构建了两自由度弱非线性耦合的实际力学系统, 然后用近似 Lie 对称性理论研究了该系统的一阶近似 Lie 对称性与近似守恒量, 最后得到了系统的 6 个一阶近似 Lie 对称性与近似守恒量, 6 个近似守恒量中 1 个为精确守恒量, 4 个为平凡的一阶近似守恒量, 只有 1 个为稳定的一阶近似守恒量. 本文的研究进一步拓展了近似对称性的研究, 同时为近似对称性理论推广应用于实际力学系统提供了一有效的途经.

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The first order approximate Lie symmetries and approximate conserved quantities of the weak nonlinear coupled two-dimensional system*

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Abstract

A real weak nonlinear coupled two-dimensional system is constructed first by using coupling spring with variable force constant. The first-order approximate Lie symmetries and approximate conserved quantities of the system are studied. The system possesses six first-order approximate Lie symmetries and approximate conserved quantities, of which one is an exact conserved quantity, four are trivial conserved quantities, and only one is a stable conserved quantity.

Keywords: weak nonlinear coupled two-dimensional system, approximate Lie symmetries, approximate conserved quantity

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