涉及双变数Hermite多项式的二项式定理及其在量子光学中的应用*

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我们用量子力学算符Hermite多项式方法发现了涉及双变数Hermite多项式的二项式的两个定理, 它们在计算若干量子光场的物理性质时有实质性的应用. 该方法不但具有简捷的优点, 而且能导出很多新的算符恒等式, 成为发展数学物理理论的一个重要分支. 若干常用的特殊函数的宗量也由普通数变为算符.

关键词: 算符厄密多项式方法, 双变数厄密多项式, 算符恒定式

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1 引 言

Hermite多项式 $H_m(x)$ 在量子力学计算谐振子能态的波函数方面有实质的应用, 它也是分数Fourier变换的本征函数, 在数理方程和概率论中Hermite 多项式也是经常出现. 在文献[1]中我们更指出从Hermite多项式可以导出拉盖尔多项式的定义, 而先前的文献都以为两者是独立的. 在文献[2]中, 我们用算符厄密多项式方法导出了涉及单变数Hermite多项式的二项式定理

$$\sum_{l=0}^{\infty}\binom{m}{l}y^{m-l}q^{l}H_{l}(x)=q^{m}H_{m}\left(x+\frac{y}{2q}\right),\tag{1}$$

这是通常的二项式定理

$$\sum_{l=0}^{m}\binom{m}{l}y^{l}q^{m-l}=(y+q)^{m}\tag{2}$$

有意义的推广. 一个有趣的问题是: 由于研究量子纠缠理论的需要, 如今常要用到双变数Hermite多项式(它不是两个独立的单变数厄密多项式的直积), 其定义是[3–6]

$$H_{m,n}(\xi,\xi^{*})=\sum_{l=0}^{\min(m,n)}\frac{m!n!}{l!(m-l)!(n-l)!}(-1)^{l}\xi^{m-l}\xi^{*n-l},\tag{3}$$

其生成函数是

$$\sum_{m,n=0}^{\infty}\frac{z^{m}z'^{n}}{m!n!}H_{m,n}(\xi,\xi^{*})=\exp\{-zz'+z\xi+z'\xi^{*}\}.\tag{4}$$

例如两粒子纠缠态[7]它可以构成一个表象

$$\langle\xi|=\langle 00|\exp\left(-|\xi|^{2}/2-ab+a\xi^{*}+b\xi\right)\tag{5}$$

就可以用(4)在Fock空间展开为

$$\langle\xi|=\mathrm{e}^{-|\xi|^{2}/2}\langle 00|\sum_{m,n=0}\frac{a^{m}b^{n}}{m!n!}H_{m,n}(\xi,\xi^{*}),\tag{6}$$

这里 $[a,a^{\dagger}]=[b,b^{\dagger}]=1$, $\langle 00|$ 是双模真空态, $|00\rangle\langle 00|$ 的正规乘积展开是: $\mathrm{e}^{-a^{\dagger}a-b^{\dagger}b}$:, 于是可得双模粒子数态 $|m,n\rangle$ 在 $\langle\xi|$ 表象中的波函数是

$$\langle\xi|m,n\rangle=\frac{1}{\sqrt{m!n!}}H_{m,n}(\xi,\xi^{*})\,\mathrm{e}^{-|\xi|^{2}/2}.\tag{7}$$

与 $|\xi\rangle$ 共轭的态

$$|\eta\rangle=\exp\left[-\frac{|\eta|^{2}}{2}+\eta a^{\dagger}-\eta^{*}b^{\dagger}+a^{\dagger}b\right]|00\rangle$$

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也是一个纠缠态, 它是两粒子相对坐标与总动量的共同本征态, 所以恰好Einstein等[8]在1935年提出的连续变量态的量子纠缠, 这方面的工作可见文献[9].

那么是否有涉及双变数Hermite多项式$H_{m,n}(\xi,\xi^*)$的广义二项式定理呢? 答案是肯定的. 以下我们将推导出新定理1

$$\begin{aligned}&\sum_{l=0}^{m}\binom{m}{l}\tau^{m-l}q^{l}H_{m-l,l}\left(\xi,\xi^*\right)\\&=(\tau q)^{m/2}H_m\left(\frac{\tau\xi+q\xi^*}{2\sqrt{\tau q}}\right),\end{aligned}\tag{8}$$

其中H_m是单变数Hermite多项式; 和新定理2

$$\begin{aligned}&\sum_{l=0}^{m}\binom{m}{l}f^{l}\sum_{k=0}^{n}\binom{n}{k}g^{k}H_{m-l,n-k}\left(x,y\right)\\&=H_{m,n}\left(f+x,g+y\right),\end{aligned}\tag{9}$$

并给出其物理应用.

2 涉及双变数厄密多项式的新二项式定理1

(8)式的左边如何求和呢? 为此, 我们用算符厄密多项式方法, 先讨论

$$\sum_{l=0}^{m}\binom{m}{l}\tau^{m-l}q^{l}H_{m-l,l}\left(a+b^{\dagger},a^{\dagger}+b\right)$$

的求和. 回忆(5)式中纠缠态$\langle\xi|$的性质, $\langle\xi|$满足本征方程[7,10]

$$\left(a+b^{\dagger}\right)|\xi\rangle=\xi|\xi\rangle,\left(a^{\dagger}+b\right)|\xi\rangle=\xi^*|\xi\rangle,\tag{10}$$

及完备性

$$\begin{aligned}&\int\frac{\mathrm{d}^2\xi}{\pi}|\xi\rangle\langle\xi|\\&=\int\frac{\mathrm{d}^2\xi}{\pi}:\mathrm{e}^{-\left(\xi-a-b^{\dagger}\right)\left(\xi^*-a^{\dagger}-b\right)}:=1,\end{aligned}\tag{11}$$

和正交性

$$\begin{aligned}\langle\xi'|\xi\rangle&=\pi\delta^{(2)}\left(\xi-\xi'\right)\\&\equiv\pi\delta\left(\xi-\xi'\right)\delta\left(\xi^*-\xi'^*\right).\end{aligned}\tag{12}$$

用算符厄密多项式方法考虑$H_{m,n}\left(a+b^{\dagger},a^{\dagger}+b\right)$的正规乘积展开, 它可以用有序算符内的积分技术(IWOP技术)[11−14]得到

$$\begin{aligned}&H_{m,n}\left(a+b^{\dagger},a^{\dagger}+b\right)\\&=\int\frac{\mathrm{d}^2\xi}{\pi}H_{m,n}\left(\xi,\xi^*\right)|\xi\rangle\langle\xi|\\&=\int\frac{\mathrm{d}^2\xi}{\pi}H_{m,n}\left(\xi,\xi^*\right):\mathrm{e}^{-\left(\xi-a-b^{\dagger}\right)\left(\xi^*-a^{\dagger}-b\right)}:\\&=:\left(a+b^{\dagger}\right)^m\left(a^{\dagger}+b\right)^n:,\end{aligned}\tag{13}$$

这是一个算符恒等式, 把它代入下式得到:

$$\begin{aligned}&\sum_{l=0}^{m}\binom{m}{l}\tau^{m-l}q^{l}H_{m-l,l}\left(a+b^{\dagger},a^{\dagger}+b\right)\\&=\sum_{l=0}^{m}\frac{m!\tau^{m-l}}{l!\,(m-l)!}:\left(a+b^{\dagger}\right)^{m-l}q^{l}\left(a^{\dagger}+b\right)^{l}:\\&=:\left[\tau\left(a+b^{\dagger}\right)+q\left(a^{\dagger}+b\right)\right]^m:.\end{aligned}\tag{14}$$

为了进一步"脱去": $\left[\tau\left(a+b^{\dagger}\right)+q\left(a^{\dagger}+b\right)\right]^m$:两边的正规乘积记号, 我们用单变数厄密多项式H_n的母函数公式

$$\sum_{n=0}\frac{t^n}{n!}H_n\left(x\right)=\exp\left[-t^2+2tx\right].\tag{15}$$

考虑

$$\begin{aligned}&\sum_{n=0}\frac{t^n}{n!}H_n\left[\frac{\tau\left(a+b^{\dagger}\right)+q\left(a^{\dagger}+b\right)}{2\sqrt{\tau q}}\right]\\&=\exp\left[-t^2+t\frac{\tau\left(a+b^{\dagger}\right)+q\left(a^{\dagger}+b\right)}{\sqrt{\tau q}}\right]\\&=:\exp\left[t\frac{\tau\left(a+b^{\dagger}\right)+q\left(a^{\dagger}+b\right)}{\sqrt{\tau q}}\right]:\\&=\sum_{n=0}\frac{t^n}{n!}:\left[\frac{\tau\left(a+b^{\dagger}\right)+q\left(a^{\dagger}+b\right)}{\sqrt{\tau q}}\right]^n:,\end{aligned}\tag{16}$$

比较两边t^n的系数得到

$$\begin{aligned}&H_n\left[\frac{\tau\left(a+b^{\dagger}\right)+q\left(a^{\dagger}+b\right)}{2\sqrt{\tau q}}\right]\\&=:\left[\frac{\tau\left(a+b^{\dagger}\right)+q\left(a^{\dagger}+b\right)}{\sqrt{\tau q}}\right]^n:.\end{aligned}\tag{17}$$

由此可见(14)式变为

$$\begin{aligned}&\sum_{l=0}^{m}\binom{m}{l}\tau^{m-l}q^{l}H_{m-l,l}\left(a+b^{\dagger},a^{\dagger}+b\right)\\&=\left(\sqrt{\tau q}\right)^m H_m\left[\frac{\tau\left(a+b^{\dagger}\right)+q\left(a^{\dagger}+b\right)}{2\sqrt{\tau q}}\right].\end{aligned}\tag{18}$$

由于$\left[a+b^{\dagger},a^{\dagger}+b\right]=0$, 所以在回到经典情形时可以将$\left(a+b^{\dagger},a^{\dagger}+b\right)\to\left(\xi,\xi^*\right)$, 这样我们有(8)式成立, 这就证明了涉及双变数Hermite多项式的二项式定理1.

特别, 当$q=1$, (8)式变为

$$\sum_{l=0}^{m}\binom{m}{l}\tau^{m-l}H_{m-l,l}\left(\xi,\xi^*\right)$$

$$=\tau^{m/2}H_m\left(\frac{\tau\xi+\xi^*}{2\sqrt{\tau}}\right). \tag{19}$$

当 $\tau=1$, 有

$$\sum_{l=0}^{m}\binom{m}{l}H_{m-l,l}\left(\xi,\xi^*\right)=H_m\left(\frac{\xi+\xi^*}{2}\right), \tag{20}$$

所以

$$\sum_{l=0}^{m}\binom{m}{l}H_{m-l,l}\left(x,x\right)=H_m\left(x\right). \tag{21}$$

3 涉及双变数厄密多项式的新二项式定理2

我们进一步证明 (9) 式中的新二项式定理2. 用算符Hermite多项式方法和 (4) 式我们有

$$\begin{aligned}&\sum_{n,m=0}^{\infty}\frac{\tau^n t^m}{n!m!}a^{\dagger n}a^m\\&=\mathrm{e}^{\tau a^\dagger}\mathrm{e}^{ta}=\mathrm{e}^{-t\tau}\mathrm{e}^{ta}\mathrm{e}^{\tau a^\dagger}\\&=\vdots\exp\left(-t\tau+ta+\tau a^\dagger\right)\vdots\\&=\sum_{n,m=0}^{\infty}\frac{\tau^n t^m}{n!m!}\vdots H_{m,n}\left(a,a^\dagger\right)\vdots,\end{aligned} \tag{22}$$

这里记号 $\vdots\ \vdots$ 代表反正规乘积, 在 $\vdots\ \vdots$ 内部的产生算符与湮没算符也是可交换的. 比较 (22) 式的两边得到 $a^{\dagger n}a^m$ 的反正规乘积展开的简洁式 [12]

$$a^{\dagger n}a^m=\vdots H_{m,n}\left(a,a^\dagger\right)\vdots=\vdots H_{n,m}\left(a^\dagger,a\right)\vdots. \tag{23}$$

现在我们转而考虑

$$\begin{aligned}&\sum_{l=0}^{m}\binom{m}{l}\tau^{m-l}q^l\sum_{k=0}^{n}\binom{n}{k}\sigma^{n-k}p^k\vdots H_{m-l,n-k}\left(a^\dagger,a\right)\vdots\\&=\sum_{l=0}^{m}\binom{m}{l}\tau^{m-l}q^l\sum_{k=0}^{n}\binom{n}{k}\sigma^{n-k}p^k:a^{\dagger m-l}a^{n-k}:\\&=:\left(q+\tau a^\dagger\right)^m\left(p+\sigma a\right)^n:\end{aligned} \tag{24}$$

两边乘 $\sum_{m,n}\frac{t^m s^n}{m!n!}$ 求和, 在其右边用 (4) 式得到

$$\begin{aligned}&\sum_{m,n}^{\infty}\frac{t^m s^n}{m!n!}:\left(q+\tau a^\dagger\right)^m\left(p+\sigma a\right)^n:\\&=:\mathrm{e}^{t\left(q+\tau a^\dagger\right)}\mathrm{e}^{s\left(p+\sigma a\right)}\\&=\vdots\mathrm{e}^{s\left(p+\sigma a\right)}\mathrm{e}^{t\left(q+\tau a^\dagger\right)-t\tau s\sigma}\vdots\\&=\vdots\mathrm{e}^{\sigma s\left(\frac{p}{\sigma}+a\right)+t\tau\left(\frac{q}{\tau}+a^\dagger\right)-(t\tau)(s\sigma)}\vdots\\&=\sum_{m,n}\frac{(t\tau)^m(s\sigma)^n}{m!n!}\vdots H_{m,n}\left(\frac{q}{\tau}+a^\dagger,\frac{p}{\sigma}+a\right)\vdots.\end{aligned} \tag{25}$$

故

$$\begin{aligned}&\sum_{l=0}^{m}\binom{m}{l}\tau^{m-l}q^l\sum_{k=0}^{n}\binom{n}{k}\sigma^{n-k}p^k\vdots H_{m-l,n-k}\left(a^\dagger,a\right)\vdots\\&=\tau^m\sigma^n\vdots H_{m,n}\left(\frac{q}{\tau}+a^\dagger,\frac{p}{\sigma}+a\right)\vdots.\end{aligned} \tag{26}$$

于是

$$\begin{aligned}&\sum_{l=0}^{m}\binom{m}{l}\left(\frac{q}{\tau}\right)^l\sum_{k=0}^{n}\binom{n}{k}\left(\frac{p}{\sigma}\right)^k H_{m-l,n-k}\left(x,y\right)\\&=H_{m,n}\left(\frac{q}{\tau}+x,\frac{p}{\sigma}+y\right).\end{aligned} \tag{27}$$

这就证明了新二项式定理2. 或可将 (27) 式写为

$$\begin{aligned}&\sum_{l=0}^{m}\binom{m}{l}\tau^l q^{m-l}\sum_{k=0}^{n}\binom{n}{k}\sigma^k p^{n-k}H_{l,k}\left(x,y\right)\\&=\tau^m\sigma^n H_{m,n}\left(\frac{q}{\tau}+x,\frac{p}{\sigma}+y\right).\end{aligned} \tag{28}$$

4 新二项式定理2的应用

作为此新二项式定理应用, 考虑量子光学中一个平移压缩态 $\mathrm{e}^{fa^\dagger b^\dagger+ga^\dagger+sb^\dagger}\left|00\right\rangle$ 被湮没 m 个 $a-$ 模光子和 n 个 $b-$ 模光子以后的变化情形, 为此需将 $a^m b^n\,\mathrm{e}^{fa^\dagger b^\dagger+ga^\dagger+sb^\dagger}$ 化为正规乘积, 用Baker-Hausdorff公式得

$$\begin{aligned}&a^m b^n\,\mathrm{e}^{fa^\dagger b^\dagger+ga^\dagger+sb^\dagger}\\&=\mathrm{e}^{fa^\dagger b^\dagger+ga^\dagger+sb^\dagger}\left(a+fb^\dagger+g\right)^m\\&\times\left(b+fa^\dagger+s\right)^n.\end{aligned} \tag{29}$$

考虑到 $\left[\left(a+fb^\dagger\right),\left(fa^\dagger+b\right)\right]=0$, 故有

$$\begin{aligned}&\mathrm{e}^{t\left(a+fb^\dagger\right)+t'\left(fa^\dagger+b\right)}\\&=\sum_{m,n=0}^{\infty}\frac{t^m t'^n}{m!n!}\left(a+fb^\dagger\right)^m\left(fa^\dagger+b\right)^n.\end{aligned} \tag{30}$$

另一方面, 用Baker-Hausdorff公式和 (4) 式可得

$$\begin{aligned}&\mathrm{e}^{t\left(a+fb^\dagger\right)+t'\left(fa^\dagger+b\right)}\\&=:\mathrm{e}^{t\left(a+fb^\dagger\right)+t'\left(fa^\dagger+b\right)}:\mathrm{e}^{ftt'}\\&=:\mathrm{e}^{\frac{a+fb^\dagger}{\sqrt{\mathrm{i}f}}\sqrt{\mathrm{i}f}t+\frac{fa^\dagger+b}{\sqrt{\mathrm{i}f}}\sqrt{\mathrm{i}f}t'}:\mathrm{e}^{ftt'}\\&=\sum_{m,n=0}^{\infty}\frac{t^m t'^n}{m!n!}\sqrt{(\mathrm{i}f)^{m+n}}\\&\quad:H_{m,n}\left[\frac{a+fb^\dagger}{\sqrt{\mathrm{i}f}},\frac{fa^\dagger+b}{\sqrt{\mathrm{i}f}}\right]:.\end{aligned} \tag{31}$$

比较 (30) 和 (31) 式得到

$$\left(a+fb^\dagger\right)^m\left(fa^\dagger+b\right)^n$$

$$=\sqrt{(\mathrm{i}f)^{m+n}}:H_{m,n}\left[\frac{a+fb^{\dagger}}{\sqrt{\mathrm{i}f}},\frac{fa^{\dagger}+b}{\sqrt{\mathrm{i}f}}\right]:. \tag{32}$$

把它代入下式

$$\begin{aligned}&\left(a+fb^{\dagger}+g\right)^{m}\left(b+fa^{\dagger}+s\right)^{n}\\=&\sum_{l=0}^{m}\binom{m}{l}g^{m-l}\left(a+fb^{\dagger}\right)^{l}\\&\times\sum_{k=0}^{n}\binom{n}{k}s^{n-k}\left(b+fa^{\dagger}\right)^{k}\\=&\sum_{l=0}^{m}\binom{m}{l}g^{m-l}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}\sqrt{(\mathrm{i}f)^{l+k}}\\&:H_{l,k}\left(\frac{a+fb^{\dagger}}{\sqrt{\mathrm{i}f}},\frac{fa^{\dagger}+b}{\sqrt{\mathrm{i}f}}\right):\\=&\sqrt{(\mathrm{i}f)^{m+n}}\\&\times:H_{l,k}\left(\frac{g+a+fb^{\dagger}}{\sqrt{\mathrm{i}f}},\frac{p+fa^{\dagger}+b}{\sqrt{\mathrm{i}f}}\right):,\end{aligned} \tag{33}$$

于是就得

$$\begin{aligned}&a^{m}b^{n}\,\mathrm{e}^{fa^{\dagger}b^{\dagger}+ga^{\dagger}+sb^{\dagger}}\left|00\right\rangle\\=&\mathrm{e}^{fa^{\dagger}b^{\dagger}+ga^{\dagger}+sb^{\dagger}}\left(a+fb^{\dagger}+g\right)^{m}\\&\times\left(b+fa^{\dagger}+s\right)^{n}\left|00\right\rangle\\=&\sqrt{(\mathrm{i}f)^{m+n}}\,\mathrm{e}^{fa^{\dagger}b^{\dagger}+ga^{\dagger}+sb^{\dagger}}\\&\times:H_{l,k}\left(\frac{g+a+fb^{\dagger}}{\sqrt{\mathrm{i}f}},\frac{p+fa^{\dagger}+b}{\sqrt{\mathrm{i}f}}\right):\left|00\right\rangle\\=&\sqrt{(\mathrm{i}f)^{m+n}}H_{l,k}\left(\frac{g+fb^{\dagger}}{\sqrt{\mathrm{i}f}},\frac{p+fa^{\dagger}}{\sqrt{\mathrm{i}f}}\right)\\&\times\ \mathrm{e}^{fa^{\dagger}b^{\dagger}+ga^{\dagger}+sb^{\dagger}}\left|00\right\rangle.\end{aligned} \tag{34}$$

可见一个平移压缩态 $\mathrm{e}^{fa^{\dagger}b^{\dagger}+ga^{\dagger}+sb^{\dagger}}\left|00\right\rangle$ 被湮没 m 个 $a-$ 模光子和 n 个 $b-$ 模光子的效果相当于一个平移压缩态受双模Hermite多项式算符的激发.

5　新二项式定理1对研究原子相干态的应用

原子相干态定义(也被称为角动量相干态或自旋相干态)为[15]

$$\exp\{\xi J_{+}-\xi^{*}J_{-}\}\left|j,-j\right\rangle,\xi=\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi}, \tag{35}$$

这里 j 是角动量或自旋量子数, 是整数或半整数, J_{+} 是角动量态 $\left|j,m\right\rangle$ 的上升算符, $\left|j,m\right\rangle$ 是 J_z 和 J^2 的共同本征态, $\left|j,-j\right\rangle$ 是最低权态, 由 J_{-} 湮没. 用 $[J_{+},J_{-}]=2J_z$ 可以将 (35) 式化为

$$\left(1+|\tau|^{2}\right)^{-j}\mathrm{e}^{\tau J_{+}}\left|j,-j\right\rangle=\left|\tau\right\rangle,$$

$$\tau=\mathrm{e}^{-\mathrm{i}\varphi}\tan\frac{\theta}{2}. \tag{36}$$

$\left|\tau\right\rangle$ 在 j 子空间有完备性

$$\begin{aligned}&\int\frac{\mathrm{d}\Omega}{4\pi}\left|\tau\right\rangle\left\langle\tau\right|=\sum_{m=-j}^{j}\left|j,m\right\rangle\left\langle j,m\right|=1,\\&\mathrm{d}\Omega=\sin\theta\mathrm{d}\theta\mathrm{d}\varphi.\end{aligned} \tag{37}$$

用Schwinger的角动量玻色实现[16]

$$\begin{aligned}J_{+}&=a^{\dagger}a-b^{\dagger}b,\\J_{-}&=b^{\dagger}a,\\J_{z}&=\frac{1}{2}\left(a^{\dagger}a-b^{\dagger}b\right),\end{aligned} \tag{38}$$

这里 $\left[a,a^{\dagger}\right]=\left[b,b^{\dagger}\right]=1$, J_z 和 J^2 的共同本征态 $\left|j,m\right\rangle$ 在双模Fock空间中为

$$\begin{aligned}\left|j,m\right\rangle=&\frac{a_{1}^{\dagger j+m}b^{\dagger j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|00\right\rangle\\=&\left|j+m\right\rangle_{a}\otimes\left|j-m\right\rangle_{b},\\\left|j,-j\right\rangle=&\left|0\right\rangle\otimes\left|2j\right\rangle,\end{aligned} \tag{39}$$

这里 $\left|j+m\right\rangle_{a}\otimes\left|j-m\right\rangle_{b}$ 是双模Fock空间中的基矢量. 用

$$\begin{aligned}&\mathrm{e}^{\xi J_{+}-\xi^{*}J_{-}}b^{\dagger}\,\mathrm{e}^{-\xi J_{+}+\xi^{*}J_{-}}\\=&b^{\dagger}\cos\frac{\theta}{2}+a^{\dagger}\,\mathrm{e}^{-\mathrm{i}\varphi}\sin\frac{\theta}{2},\end{aligned} \tag{40}$$

并注意到

$$\mathrm{e}^{\xi J_{+}-\xi^{*}J_{-}}\left|00\right\rangle=\left|00\right\rangle, \tag{41}$$

在Schwinger的角动量玻色实现下 $\left|\tau\right\rangle$ 表达为

$$\begin{aligned}\left|\tau\right\rangle=&\mathrm{e}^{\xi J_{+}-\xi^{*}J_{-}}\left|0\right\rangle_{a}\otimes\left|2j\right\rangle_{b}\\=&\mathrm{e}^{\xi J_{+}-\xi^{*}J_{-}}\frac{b^{\dagger 2j}}{\sqrt{(2j)!}}\mathrm{e}^{-(\xi J_{+}-\xi^{*}J_{-})}\\&\times\ \mathrm{e}^{\xi J_{+}-\xi^{*}J_{-}}\left|0\right\rangle_{a}\otimes\left|0\right\rangle_{b}\\=&\frac{1}{\sqrt{(2j)!}}\left(b^{\dagger}\cos\frac{\theta}{2}+a^{\dagger}\,\mathrm{e}^{-\mathrm{i}\varphi}\sin\frac{\theta}{2}\right)^{2j}\left|00\right\rangle\\=&\sum_{l=0}^{2j}\left[\frac{(2j)!}{l!(2j-l)!}\right]^{1/2}\left(\cos\frac{\theta}{2}\right)^{l}\\&\times\left(\mathrm{e}^{-\mathrm{i}\varphi}\sin\frac{\theta}{2}\right)^{2j-l}\left|2j-l\right\rangle\otimes\left|l\right\rangle\\=&\frac{1}{(1+|\tau|^{2})^{j}}\sum_{l=0}^{2j}\left[\frac{(2j)!}{l!\,(2j-l)!}\right]^{1/2}\\&\times\tau^{2j-l}\left|2j-l\right\rangle\otimes\left|l\right\rangle.\end{aligned} \tag{42}$$

这可以被看做是$|\tau\rangle$的Schmidt分解, 所以$|\tau\rangle$是一个纠缠态. 由(6)式得双模粒子数态在纠缠态表象中的波函数是

$$\langle\xi|\, m,n\rangle = \mathrm{e}^{-|\xi|^2/2}H_{m,n}(\xi,\xi^*)/\sqrt{m!n!}. \tag{43}$$

所以

$$\begin{aligned}&\langle\xi|\,(|2j-l\rangle\otimes|l\rangle)\\ =&\frac{\mathrm{e}^{-|\xi|^2/2}}{\sqrt{l!(2j-l)!}}H_{2j-l,l}(\xi,\xi^*).\end{aligned} \tag{44}$$

把(44)代入(42)式并用广义二项式定理(8)立刻得到

$$\begin{aligned}\langle\xi\,|\tau\rangle =&\frac{1}{(1+|\tau|^2)^j}\sum_{l=0}^{2j}\left[\frac{(2j)!}{l!\,(2j-l)!}\right]^{1/2}\\ &\times\tau^{2j-l}\frac{\mathrm{e}^{-|\xi|^2/2}}{\sqrt{l!(2j-l)!}}H_{2j-l,l}(\xi,\xi^*)\\ =&\,\mathrm{e}^{-|\xi|^2/2}\frac{\sqrt{(2j)!}}{(1+|\tau|^2)^j}\sum_{l=0}^{2j}\frac{1}{l!(2j-l)!}\\ &\times\tau^{2j-l}H_{2j-l,l}(\xi,\xi^*)\\ =&\,\mathrm{e}^{-\xi^*\xi/2}\frac{\tau^j}{\sqrt{(2j)!}\,(1+|\tau|^2)^j}\\ &\times H_{2j}\left(\frac{\tau\xi+\xi^*}{2\sqrt{\tau}}\right).\end{aligned} \tag{45}$$

所以$|\tau\rangle$对应于原子相干态的波函数是一个单变数厄密多项式, 这与文献[17]得到的结果相同.

用量子力学算符Hermite多项式方法我们导出了涉及双变数Hermite多项式的二项式定理, 该方法不但具有简捷的优点, 可以系统地发展数学物理理论, 能导出很多新的算符恒等式. 由于双变数Hermite多项式的不同于两个单变数Hermite多项式的直积, 就像双模压缩态不是两个单模压缩态的直积, 所以有必要专门撰文予以讨论. 我们相信以算符为特殊函数宗量的理论将在发展数学物理方程方面起到重要的作用.

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Binomial theorems related to two-variable Hermite polynomials and its application in quantum optics*

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Abstract

We propose an operator Hermite polynomial method, namely, to replace the special functions' argument by quantum mechanical operator, and in this way we have derived two binomial theorems related to two-variable Hermite polynomials. This method is concise and may be of help in deducing many operator identities, which may become a new branch in mathematical physics theory.

Keywords: operator Hermite polynomial method, two-variable Hermite polynomials, operator identities

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