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Conformal invariance and conserved quantity of Mei symmetry for Appell equation in a holonomic system in relative motion

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相对运动完整系统Appell方程Mei对称性的共形不变性与守恒量*

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研究相对运动完整系统Appell方程Mei对称性的共形不变性与守恒量. 引入无限小单参数变换群及其生成元向量, 给出相对运动完整系统Appell方程的Mei对称性和共形不变性的定义, 导出系统Mei对称性的共形不变性确定方程, 重点讨论系统共形不变性和Mei对称性的关系, 然后借助规范函数满足的结构方程导出系统Mei对称性导致的Mei守恒量表达式, 最后举例说明结果的应用.

关键词: 相对运动, 完整系统, Appell方程, 共形不变性

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1 引 言

随着近代科学技术的发展, 研究复杂力学系统已成为分析力学的一个发展方向. 力学系统的对称性理论主要有 Noether 对称性, Lie 对称性和 Mei 对称性, 得到的守恒量主要有 Noether 守恒量, Hojman 守恒量和Mei 守恒量, 此外还有混合形式, 利用对称性寻找守恒量的研究已经取得了丰硕成果[1−21]. 利用共形不变性理论在寻求复杂力学系统对称性及守恒量方面也已取得了一些成果[22−26].

用分析力学的方法研究复杂力学系统的运动时, 在动坐标系中的研究称为相对运动动力学, 目前复杂力学系统的相对运动动力学问题已成为科学研究和生产实际中一个重要的领域. 一些学者研究了相对运动动力学系统Lagrange方程的对称性和守恒量[27−30]; 在Appell方程方面, Zhang等[31]研究了变质量完整约束相对运动动力学系统Appell方程的Mei对称性与守恒量, Xie等[32]研究了相对运动动力学系统Appell方程的特殊的Lie对称性与Hojman守恒量, 但现有文献表明还没有研究相对运动完整系统Appell方程Mei对称性的共形不变性, 本文将研究相对运动完整系统Appell 方程Mei对称性的共形不变性与守恒量.

2 相对运动完整系统的Appell方程和运动微分方程

设固定于运动参考物上动坐标系原点O的速度v_0以及参考物的角速度$\boldsymbol{\omega}$为时间t的已知函数. 系统由N个质点组成, 质点位置由n个广义坐标$q_s\,(s=1,\cdots,n)$来确定. 本文采用Einstein求和约定, 则力学系统相对运动加速度能量和相对运动完整系统的Appell方程分别为

$$S_r = S_r\left(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}\right) = \frac{1}{2}m_i\ddot{\boldsymbol{r}}'_i\cdot\ddot{\boldsymbol{r}}'_i,\quad (i=1,\cdots,N),\tag{1}$$

$$\frac{\partial S_r}{\partial\ddot{q}_s} = Q_s - \frac{\partial}{\partial q_s}\left(V^o+V^\omega\right) + Q_s^{\dot\omega} + \varGamma_s,$$

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$$(s=1,\cdots,n). \tag{2}$$

其中Q_s为广义力, $V^o = M(v_0^* + \boldsymbol{\omega}\times v_0)\cdot \boldsymbol{r'}_{\mathrm{c}}$为均匀力场势能, $V^\omega = -\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\theta}^o\cdot\boldsymbol{\omega}$为离心力势能, $Q_s^{\dot{\boldsymbol{\omega}}} = -\sum\limits_{i=1}^{N} m_i(\dot{\omega}\times\boldsymbol{r'}_i)\cdot\frac{\partial \boldsymbol{r'}_i}{\partial q_s}$为广义回转惯性力, $\varGamma_s = 2\boldsymbol{\omega}\cdot\left(m_i\frac{\partial \boldsymbol{r'}_i}{\partial q_s}\times\frac{\partial \boldsymbol{r'}_i}{\partial q_k}\right)\dot{q}_k$为广义陀螺力.

利用方程(2)式可解出所有广义加速度——系统的运动微分方程

$$\ddot{q}_s = \alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\quad (s=1,2,\cdots,n). \tag{3}$$

3 系统Appell方程Mei对称性的共形不变性

引入时间和广义坐标的无限小变换的展开式

$$\begin{aligned}
&t^* = t+\varepsilon\xi_0\left(t,\boldsymbol{q},\dot{\boldsymbol{q}}\right),\\
&q_s^*\left(t^*\right) = q_s\left(t\right)+\varepsilon\xi_s\left(t,\boldsymbol{q},\dot{\boldsymbol{q}}\right),\\
&(s=1,2,\cdots,n),
\end{aligned}\tag{4}$$

其中ε为无限小参数, ξ_0,ξ_s为无限小变换生成元. 由(4)式可得

$$\begin{aligned}
\frac{\mathrm{d}q_s^*}{\mathrm{d}t^*} &= \frac{\mathrm{d}q_s+\varepsilon\mathrm{d}\xi_s}{\mathrm{d}t+\varepsilon\mathrm{d}\xi_0}\\
&= \dot{q}_s+\varepsilon\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)+O\left(\varepsilon^2\right),\\
\frac{\mathrm{d}^2q_s^*}{\mathrm{d}t^{*2}} &= \ddot{q}_s+\varepsilon\left[\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)^{\cdot}-\ddot{q}_s\dot{\xi}_0\right]\\
&\quad +O\left(\varepsilon^2\right).
\end{aligned}\tag{5}$$

假设在经历无限小变换(4)式后, 系统的动力学函数S_r, Q_s, V^o, V^ω, $Q_s^{\dot{\omega}}$和$\varGamma_s$, 分别变为S_r^*, Q_s^*, V^{o^*}, $V^{\omega *}$, $Q_s^{\dot{\omega}*}$和$\varGamma_s^*$, 将S_r^*, Q_s^*, V^{o^*}, $V^{\omega *}$, $Q_s^{\dot{\omega}*}$和$\varGamma_s^*$在$(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$处作Taylor级数展开, 有

$$\begin{aligned}
S_r^* &= S_r\left(t^*,\boldsymbol{q}^*,\frac{\mathrm{d}q^*}{\mathrm{d}t^*},\frac{\mathrm{d}^2q^*}{\mathrm{d}t^{*2}}\right)\\
&= S_r(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})+\varepsilon\left\{\frac{\partial S_r}{\partial t}\xi_0+\frac{\partial S_r}{\partial q_s}\xi_s\right.\\
&\quad+\frac{\partial S_r}{\partial \dot{q}_s}(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\\
&\quad\left.+\frac{\partial S_r}{\partial \ddot{q}_s}[(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)^{\cdot}-\ddot{q}_s\dot{\xi}_0]\right\}+O(\varepsilon^2),
\end{aligned}$$

即

$$S_r^* = S_r(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})+\varepsilon\tilde{\boldsymbol{X}}^{(2)}(S_r)+O(\varepsilon^2), \tag{6}$$

$$\begin{aligned}
&Q_s^* = Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\tilde{\boldsymbol{X}}^{(1)}(Q_s)+O(\varepsilon^2),\\
&(s=1,2,\cdots,n),
\end{aligned}\tag{7}$$

$$V^{o^*} = V^o(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\tilde{\boldsymbol{X}}^{(1)}(V^o)+O(\varepsilon^2), \tag{8}$$

$$V^{\omega *} = V^\omega(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\tilde{\boldsymbol{X}}^{(1)}(V^\omega)+O(\varepsilon^2), \tag{9}$$

$$\begin{aligned}
&Q_s^{\dot{\omega}*} = Q_s^{\dot{\omega}}(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\tilde{\boldsymbol{X}}^{(1)}(Q_s^{\dot{\omega}})+O(\varepsilon^2),\\
&(s=1,2,\cdots,n),
\end{aligned}\tag{10}$$

$$\begin{aligned}
&\varGamma_s^* = \varGamma_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\tilde{\boldsymbol{X}}^{(1)}(\varGamma_s)+O(\varepsilon^2),\\
&(s=1,2,\cdots,n).
\end{aligned}\tag{11}$$

其中

$$\begin{aligned}
\tilde{\boldsymbol{X}}^{(2)} &= \tilde{\boldsymbol{X}}^{(1)}+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}-\dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)\right.\\
&\quad\left.-\ddot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right]\frac{\partial}{\partial\ddot{q}_s},
\end{aligned}\tag{12}$$

$$\tilde{\boldsymbol{X}}^{(1)} = \boldsymbol{X}^{(0)}+\left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}-\dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)\frac{\partial}{\partial\dot{q}_s}, \tag{13}$$

$$\boldsymbol{X}^{(0)} = \xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}, \tag{14}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t}+\dot{q}_s\frac{\partial}{\partial q_s}+\alpha_s\frac{\partial}{\partial\dot{q}_s}+\dot{\alpha}_s\frac{\partial}{\partial\ddot{q}_s}. \tag{15}$$

定义1 如果用经无限小变换(4)式变换后的动力学函数S_r^*, Q_s^*, V^{o^*}, $V^{\omega *}$, $Q_s^{\dot{\omega}*}$和$\varGamma_s^*$代替变换前的动力学函数S_r, Q_s, V^o, V^ω, $Q_s^{\dot{\omega}}$和$\varGamma_s$, 相对运动完整系统的Appell方程(2)的形式保持不变, 即

$$\begin{aligned}
&\frac{\partial S_r^*}{\partial\ddot{q}_s} = Q_s^*-\frac{\partial}{\partial q_s}\left(V^{o^*}+V^{\omega *}\right)+Q_s^{\dot{\omega}*}+\varGamma_s^*,\\
&(s=1,\cdots,n).
\end{aligned}\tag{16}$$

则这种对称性称为相对运动完整系统Appell方程(2)的Mei对称性.

将(6), (7), (8), (9), (10)和(11)式代入方程(16) 式, 忽略ε^2以上的高阶小项, 并利用方程(2)可得

$$\begin{aligned}
&\frac{\partial}{\partial\ddot{q}_s}\left[\tilde{\mathrm{X}}^{(2)}\left(S_r\right)\right]\\
&-\tilde{\boldsymbol{X}}^{(1)}\left[Q_s-\frac{\partial}{\partial q_s}\left(V^o+V^\omega\right)+Q_s^{\dot{\omega}}+\varGamma_s\right]=0,\\
&(s=1,\cdots,n),
\end{aligned}\tag{17}$$

方程(17)式称为相对运动完整系统Appell方程Mei对称性的判据方程.

定义2 对于相对运动完整系统的Appell方程(2), 如果存在矩阵M_s^k满足

$$\begin{aligned}
&\frac{\partial}{\partial\ddot{q}_s}[\tilde{\boldsymbol{X}}^{(2)}(S_r)]\\
&-\tilde{\boldsymbol{X}}^{(1)}\left[Q_s-\frac{\partial}{\partial q_s}\left(V^o+V^\omega\right)+\varGamma_s+Q_s^{\dot{\omega}}\right]
\end{aligned}$$

$$
= M_s^k \left\{ \frac{\partial S_r}{\partial \ddot{q}_k} - \left[Q_k - \frac{\partial}{\partial q_k} (V^o + V^\omega) + Q_k^{\dot{\omega}} + \Gamma_k \right] \right\}, \quad (s, k = 1, 2, \cdots, n), \tag{18}
$$

则方程 (2) 在无限小变换 (4) 式的作用下具有Mei对称性的共形不变性. (18) 式是满足Mei对称性共形不变性的确定方程, 其中 M_s^k 为共形因子.

命题1　如果方程 (2) 在无限小变换 (4) 式作用下是Mei对称性的, 且存在矩阵 Γ_s^k 满足

$$
\begin{aligned}
& \frac{\partial}{\partial \ddot{q}_s} [\tilde{\boldsymbol{X}}^{(2)}(S_r)] - \tilde{\boldsymbol{X}}^{(1)} \left[Q_s - \frac{\partial}{\partial q_s} (V^o + V^\omega) + Q_s^{\dot{\omega}} + \Gamma_s \right] \\
& - \left\{ \frac{\partial}{\partial \ddot{q}_s} [\tilde{\mathrm{X}}^{(2)}(S_r)] - \tilde{\boldsymbol{X}}^{(1)} \left[Q_s - \frac{\partial}{\partial q_s} (V^o + V^\omega) + Q_s^{\dot{\omega}} + \Gamma_s \right] \right\} \Bigg|_{\frac{\partial S_r}{\partial \ddot{q}_s} = Q_s - \frac{\partial}{\partial q_s}(V^o + V^\omega) + Q_s^{\dot{\omega}} + \Gamma_s} \\
= {} & \Gamma_s^k \left(\frac{\partial S_r}{\partial \ddot{q}_k} - Q_k + \frac{\partial}{\partial q_k} (V^o + V^\omega) - Q_k^{\dot{\omega}} - \Gamma_k \right), \\
& (s, k = 1, 2, \cdots, n),
\end{aligned} \tag{19}
$$

则方程 (2) 在无限小变换 (4) 式作用下具有共形不变性, 同时又具有Mei对称性的充分与必要条件为 $M_s^k = \Gamma_s^k$.

证明　由于方程 (2) 的Mei对称性满足 (17) 式, 如果存在一个矩阵 Γ_s^k 满足 (19) 式, 则 (19) 式成为

$$
\begin{aligned}
& \frac{\partial}{\partial \ddot{q}_s} [\tilde{\boldsymbol{X}}^{(2)}(S_r)] - \tilde{\boldsymbol{X}}^{(1)} \left[Q_s - \frac{\partial}{\partial q_s} (V^o + V^\omega) + Q_s^{\dot{\omega}} + \Gamma_s \right] \\
= {} & \Gamma_s^k \left\{ \frac{\partial S_r}{\partial \ddot{q}_k} - \left[Q_k - \frac{\partial}{\partial q_k} (V^o + V^\omega) + Q_k^{\dot{\omega}} + \Gamma_k \right] \right\}.
\end{aligned} \tag{20}
$$

由定义 (18) 式, 系统的共形因子 $M_s^k = \Gamma_s^k$.

反之亦然, 由定义 (18) 式和 (19) 式, 容易验证

$$
\begin{aligned}
& (M_s^k - \Gamma_s^k) \left\{ \frac{\partial S_r}{\partial \ddot{q}_k} - \left[Q_k - \frac{\partial}{\partial q_k} (V^o + V^\omega) + Q_k^{\dot{\omega}} + \Gamma_k \right] \right\} \\
= {} & \left\{ \frac{\partial}{\partial \ddot{q}_s} [\tilde{\mathrm{X}}^{(2)}(S_r)] - \tilde{\boldsymbol{X}}^{(1)} \left[Q_k - \frac{\partial}{\partial q_k} (V^o + V^\omega) + Q_k^{\dot{\omega}} + \Gamma_k \right] \right\} \Bigg|_{\frac{\partial S_r}{\partial \ddot{q}_s} = Q_s - \frac{\partial}{\partial q_s}(V^o + V^\omega) + Q_s^{\dot{\omega}} + \Gamma_s}, \\
& (s, k = 1, 2, \cdots, n),
\end{aligned} \tag{21}
$$

若 $M_s^k = \Gamma_s^k$, 则容易得到 (17) 式, 因而系统具有Mei对称性.

4　系统Appell方程Mei对称性导致的Mei守恒量

根据相对运动完整系统Appell方程的Mei对称性理论, Mei对称性共形不变性满足一定条件时也可导致相应的守恒量, 结果如下:

命题2　如果相对运动完整系统Appell方程 (2) 的Mei对称性的生成元 ξ_0,ξ_s 以及规范函数 $G_{\mathrm{M}} = G_{\mathrm{M}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ 满足如下结构方程:

$$
\begin{aligned}
& \tilde{\boldsymbol{X}}^{(2)} (S_r) \frac{\bar{\mathrm{d}} \xi_0}{\mathrm{d}t} + \tilde{\mathrm{X}}^{(1)} \left[\tilde{\boldsymbol{X}}^{(2)} (\boldsymbol{S}_r) \right] \\
& + (\xi_s - \dot{q}_s \xi_0) \tilde{E}_s \left[\tilde{\boldsymbol{X}}^{(2)} (\boldsymbol{S}_r) \right] \\
& + \xi_0 \left\{ \tilde{\boldsymbol{X}}^{(1)} \left[Q_s - \frac{\partial}{\partial q_s} (V^o + V^\omega) + Q_s^{\dot{\omega}} + \Gamma_s \right] \right\} \\
& \times \frac{\bar{\mathrm{d}} \alpha_s}{\mathrm{d}t} + \frac{\bar{\mathrm{d}} G_{\mathrm{M}}}{\mathrm{d}t} = 0,
\end{aligned} \tag{22}
$$

则相对运动完整系统Appell方程 (2) 的Mei对称性的共形不变性导致的Mei守恒量为

$$
\begin{aligned}
I_{\mathrm{M}} = {} & \xi_0 \tilde{\boldsymbol{X}}^{(2)} (S_r) + \frac{\partial \tilde{\boldsymbol{X}}^{(2)} (\boldsymbol{S}_r)}{\partial \dot{q}_s} (\xi_s - \dot{q}_s \xi_0) \\
& + G_{\mathrm{M}} = \mathrm{const}.
\end{aligned} \tag{23}
$$

证明　利用 (15) 式, 有

$$
\begin{aligned}
\frac{\bar{\mathrm{d}} I_{\mathrm{M}}}{\mathrm{d}t} = {} & \left[\frac{\partial \tilde{\boldsymbol{X}}^{(2)}(S_r)}{\partial t} + \dot{q}_s \frac{\partial \tilde{\boldsymbol{X}}^{(2)}(S_r)}{\partial q_s} \right. \\
& \left. + \alpha_s \frac{\partial \tilde{\boldsymbol{X}}^{(2)}(S_r)}{\partial \dot{q}_s} + \frac{\partial \tilde{\boldsymbol{X}}^{(2)}(S_r)}{\partial \ddot{q}_s} \frac{\bar{\mathrm{d}} \alpha_s}{\mathrm{d}t} \right] \xi_0 \\
& + \tilde{\boldsymbol{X}}^{(2)}(S_r) \frac{\bar{\mathrm{d}} \xi_0}{\mathrm{d}t} \\
& + \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \frac{\partial \tilde{\boldsymbol{X}}^{(2)}(S_r)}{\partial \dot{q}_s} \right] (\xi_s - \dot{q}_s \xi_0) \\
& + \frac{\partial \tilde{\boldsymbol{X}}^{(2)}(S_r)}{\partial \dot{q}_s} \left(\frac{\bar{\mathrm{d}} \xi_s}{\mathrm{d}t} - \alpha_s \xi_0 - \dot{q}_s \frac{\bar{\mathrm{d}} \xi_0}{\mathrm{d}t} \right) \\
& + \frac{\bar{\mathrm{d}} G_{\mathrm{M}}}{\mathrm{d}t},
\end{aligned} \tag{24}
$$

且

$$
\begin{aligned}
& \tilde{\boldsymbol{X}}^{(1)} \left[\tilde{\boldsymbol{X}}^{(2)} (\boldsymbol{S}_r) \right] \\
= {} & \xi_0 \frac{\partial \tilde{\boldsymbol{X}}^{(2)} (\boldsymbol{S}_r)}{\partial t} + \xi_s \frac{\partial \tilde{X}^{(2)} (S_r)}{\partial q_s}
\end{aligned}
$$

$$+\left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}-\dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)\frac{\partial\tilde{\boldsymbol{X}}^{(2)}(\boldsymbol{S}_r)}{\partial\dot{q}_s},$$

则 (24) 式变为

$$\begin{aligned}\frac{\bar{\mathrm{d}}I_{\mathrm{M}}}{\mathrm{d}t}&=\tilde{\boldsymbol{X}}^{(1)}\left[\tilde{\boldsymbol{X}}^{(2)}(\boldsymbol{S}_r)\right]-\xi_s\frac{\partial\tilde{\boldsymbol{X}}^{(2)}(\boldsymbol{S}_r)}{\partial q_s}\\&\quad+\xi_0\dot{q}_s\frac{\partial\tilde{\boldsymbol{X}}^{(2)}(\boldsymbol{S}_r)}{\partial q_s}+\xi_0\frac{\partial\tilde{\mathrm{X}}^{(2)}(\boldsymbol{S}_r)}{\partial\ddot{q}_s}\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t}\\&\quad+\tilde{\mathrm{X}}^{(2)}(S_r)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\\&\quad+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\tilde{\boldsymbol{X}}^{(2)}(S_r)}{\partial\dot{q}_s}\right](\xi_s-\dot{q}_s\xi_0)+\frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t}\\&=\tilde{\boldsymbol{X}}^{(1)}\left[\tilde{\boldsymbol{X}}^{(2)}(\boldsymbol{S}_r)\right]-(\xi_s-\dot{q}_s\xi_0)\frac{\partial\tilde{\boldsymbol{X}}^{(2)}(\boldsymbol{S}_r)}{\partial q_s}\\&\quad+\xi_0\frac{\partial\tilde{\boldsymbol{X}}^{(2)}(\boldsymbol{S}_r)}{\partial\ddot{q}_s}\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t}+\tilde{\mathrm{X}}^{(2)}(S_r)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\\&\quad+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\tilde{\boldsymbol{X}}^{(2)}(\boldsymbol{S}_r)}{\partial\dot{q}_s}\right](\xi_s-\dot{q}_s\xi_0)+\frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t}.\end{aligned}$$

利用结构方程 (22) 式和确定方程 (17) 式, 有

$$\begin{aligned}\frac{\bar{\mathrm{d}}I_{\mathrm{M}}}{\mathrm{d}t}&=\tilde{\boldsymbol{X}}^{(1)}\left[\tilde{\boldsymbol{X}}^{(2)}(\boldsymbol{S}_r)\right]\\&\quad+(\xi_s-\dot{q}_s\xi_0)\tilde{E}_s\left[\tilde{\boldsymbol{X}}^{(2)}(S_r)\right]\\&\quad+\xi_0\frac{\partial\tilde{\boldsymbol{X}}^{(2)}(\boldsymbol{S}_r)}{\partial\ddot{q}_s}\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t}+\tilde{\boldsymbol{X}}^{(2)}(\boldsymbol{S}_r)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\\&\quad+\frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t}\\&=\xi_0\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t}\left\{\frac{\partial\tilde{\boldsymbol{X}}^{(2)}(\boldsymbol{S}_r)}{\partial\ddot{q}_s}\right.\\&\quad-\tilde{\boldsymbol{X}}^{(1)}\left[Q_s-\frac{\partial}{\partial q_s}(V^o+V^\omega)\right.\\&\quad\left.\left.+Q_s^{\dot{\omega}}+\varGamma_s\right]\right\}\\&=0.\end{aligned}$$

5 算 例

相对运动完整系统的加速度能量为

$$S_r=\frac{1}{2}m\left(\ddot{q}_1^2+\ddot{q}_2^2\right),\tag{25}$$

广义力为

$$Q_1=-mg\cos\alpha,\quad Q_2=-mg\sin\alpha\sin\omega t,\tag{26}$$

并有

$$V^0=Q_1^{\dot{\omega}}=Q_2^{\dot{\omega}}=\varGamma_1=\varGamma_2=0,\tag{27}$$

$$V^\omega=-\frac{1}{2}m\omega^2q_2^2.\tag{28}$$

试研究该系统 Appell 方程 Mei 对称性的共形不变性和守恒量.

将 (25), (26), (27) 和 (28) 式代入方程 (2) 可得

$$\begin{aligned}m\ddot{q}_1&=-mg\cos\alpha,\\m\ddot{q}_2&=m\left(q_2\omega^2-g\sin\alpha\sin\omega t\right).\end{aligned}\tag{29}$$

利用 (12), (13) 和 (14) 式做计算得

$$\begin{aligned}&\tilde{X}^{(2)}(S_r)\\&=m\left(\ddot{q}_1\frac{\bar{\mathrm{d}}^2\xi_1}{\mathrm{d}t^2}+\ddot{q}_2\frac{\bar{\mathrm{d}}^2\xi_2}{\mathrm{d}t^2}\right)-m\left(\dot{q}_1\ddot{q}_1+\dot{q}_2\ddot{q}_2\right)\frac{\bar{\mathrm{d}}^2\xi_0}{\mathrm{d}t^2}\\&\quad-2m\left(\ddot{q}_1^2+\ddot{q}_2^2\right)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t},\\&\frac{\partial}{\partial\ddot{q}_1}\tilde{X}^{(2)}(S_r)\\&=m\frac{\bar{\mathrm{d}}^2\xi_1}{\mathrm{d}t^2}-m\dot{q}_1\frac{\bar{\mathrm{d}}^2\xi_0}{\mathrm{d}t^2}-4m\ddot{q}_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t},\\&\frac{\partial}{\partial\ddot{q}_2}\tilde{X}^{(2)}(S_r)\\&=m\frac{\bar{\mathrm{d}}^2\xi_2}{\mathrm{d}t^2}-m\dot{q}_2\frac{\bar{\mathrm{d}}^2\xi_0}{\mathrm{d}t^2}-4m\ddot{q}_2\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t},\\&\tilde{X}^{(1)}\left[Q_1-\frac{\partial}{\partial q_1}(V^o+V^\omega)+Q_1^{\dot{\omega}}+\varGamma_1\right]=0,\\&\tilde{X}^{(1)}\left[Q_2-\frac{\partial}{\partial q_2}(V^o+V^\omega)+Q_2^{\dot{\omega}}+\varGamma_2\right]\\&=\xi_0\left(-mg\omega\sin\alpha\cos\omega t\right)+\xi_2m\omega^2.\end{aligned}$$

取无限小变换生成元

$$\begin{aligned}&\xi_0=0,\quad\xi_1=q_1+\frac{1}{2}gt^2\cos\alpha,\\&\xi_2=2q_2-\frac{1}{\omega^2}g\sin\alpha\sin\omega t.\end{aligned}\tag{30}$$

则可算得

$$\begin{aligned}&\frac{\partial}{\partial\ddot{q}_1}\tilde{X}^{(2)}(S_r)-\tilde{X}^{(1)}\left[Q_1-\frac{\partial}{\partial q_1}(V^o+V^\omega)\right.\\&\quad\left.+Q_1^{\dot{\omega}}+\varGamma_1\right]\\&=m\left(\ddot{q}_1+gcos\alpha\right),\\&\frac{\partial}{\partial\ddot{q}_2}\tilde{X}^{(2)}(S_r)-\tilde{X}^{(1)}\left[Q_2-\frac{\partial}{\partial q_2}(V^o+V^\omega)\right.\\&\quad\left.+Q_2^{\dot{\omega}}+\varGamma_2\right]\\&=2m\left(\ddot{q}_2-q_2\omega^2+g\sin\alpha\sin\omega t\right).\end{aligned}\tag{31}$$

从而由 (18), (29) 和 (31) 式得共形因子

$$\varGamma=\begin{pmatrix}1&0\\0&2\end{pmatrix}.$$

将 (29) 式代入 (31) 式得

$$\frac{\partial}{\partial\ddot{q}_1}\tilde{X}^{(2)}(S_r)-\tilde{X}^{(1)}\bigg[Q_1-\frac{\partial}{\partial q_1}(V^o+V^\omega)+Q_1^{\dot\omega}+\varGamma_1\bigg]=0,$$

$$\frac{\partial}{\partial\ddot{q}_2}\tilde{X}^{(2)}(S_r)-\tilde{X}^{(1)}\bigg[Q_2-\frac{\partial}{\partial q_2}(V^o+V^\omega)+Q_2^{\dot\omega}+\varGamma_2\bigg]=0. \quad (32)$$

从而系统满足Mei对称性. 此时, 系统既是共形不变性的又是Mei对称性的. 作计算

$$\tilde{X}^{(2)}(S_r)=m(\ddot{\xi}_2\ddot{q}_2+\ddot{\xi}_1\ddot{q}_1)=m\ddot{q}_2(\ddot{q}_2+q_2\omega^2), \quad (33)$$

$$\tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S_2)\right]=m\omega^2\xi_2\ddot{q}_2, \quad (34)$$

$$\tilde{E}_1\left[\tilde{X}^{(2)}(S_r)\right]=0, \quad (35)$$

$$\tilde{E}_2\left[\tilde{X}^{(2)}(S_r)\right]=-m\omega^2\ddot{q}_2, \quad (36)$$

将 (33)—(36) 代入结构方程 (22) 得

$$G_{\mathrm{M}}=m(\dot{q}_1+gt\cos\alpha)+m\,\mathrm{e}^{-\omega t}\bigg[\dot{q}_2+\omega q_2-\frac{g\sin\alpha}{2\omega}(\sin(\omega t)+\cos(\omega t))\bigg]=\mathrm{const}. \quad (37)$$

将 (37) 代入 (23), 得到系统Mei对称性的共形不变性直接导致的守恒量为

$$I_{\mathrm{M}}=m(\dot{q}_1+gt\cos\alpha)+m\,\mathrm{e}^{-\omega t}\bigg[\dot{q}_2+\omega q_2-\frac{g\sin\alpha}{2\omega}(\sin(\omega t)+\cos(\omega t))\bigg]=\mathrm{const}. \quad (38)$$

6 结 论

本文引入无限小单参数变换群及其生成元向量, 定义了相对运动完整系统Appell方程的Mei对称性和共形不变性, 并利用规范函数满足的结构方程导出了系统相应的Mei守恒量. 本文的结论进一步完善了Appell方程的对称性和守恒量理论.

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Conformal invariance and conserved quantity of Mei symmetry for Appell equation in a holonomic system in relative motion*

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Abstract

For a holonomic system in relative motion, the conformal invariance and the conserved quantity of Mei symmetry with Appell equations are investigated. First, by using the infinitesimal one-parameter transformation group and the infinitesimal generator vector, the definitions of Mei symmetry and the conformal invariance with Appell equations in a holonomic system in relative motion are given, and the determining equations of the conformal invariance of Mei symmetry for the system are derived. Relationship between the conformal invariance and Mei symmetry for the system is mainly studied. Then, by means of the structural equation which the gauge function satisfies, the expression of Mei conserved quantity deduced from Mei symmetry for the system is obtained. Finally, an example is given to illustrate the application of the result.

Keywords: relative motion, holonomic system, Appell equation, conformal invariance

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