THEORY OF GRAVITATION AND ELECTROMAGNETISM

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Abstract

In this paper the writer employs complex Riemannian Geometry and defines the absolute interval between two points as consisting of a real part and an imaginary part. Two postulates (I) and (II) are used: the first may be called law of gravitation and electromagnetism; the second equation of motion. In the absence of electromagnetic phenomena the theory reduces practically to Einstein's theory.

In this paper, as a first approximation, the author has obtained Newton's law of gravitation, Maxwell's laws of electromagnetism and Lorentz laws of motion. The intrinsic connections between electricity and matter can be seen at a glance but the physical concepts and the full development of the theory are hoped to be given in a later paper.

1. Einstein's successful geometrization of the gravitational field led at once to the hope that similar purely geometrical ideas might also serve completely to describe the phenomena of the electromagnetic field as embodied in the classical Maxwell-Lorentz equations and Lorentz law of force on a charged particle. This idea led Weyl and Eddington, soon after the discovery of the geometrical significance of the gravitational field, and more recently Einstein himself, to suggest modifications of Riemann's Geometry so as to make room for the electromagnetic field. None of these theories has proved quite satisfactory.[1]

The physical agent inseparably associated with gravitation is mass just as the physical entity inseparably associated with

(1) In none of these theories is the connection between the geometrical quantities interpreted as the electromagnetic potential and the charge and current densities made quite clear. In other words, the relation between these quantities and the classical retarded electromagnetic potential is not in the least apparent. Further, no purely geometrical derivation of the Lorentz law of force seems possible in any of these theories.

electromagnetism is electric charge. If a unified theory of gravitation and electromagnetism is desired it becomes necessary to examine these two fundamental entities in an endeavour to discover their analogies and differences so as to obtain the best clues for a method of attack.[2] Now the most obvious resemblance is found in the inverse square law of interaction between two masses or two charges. The difference in the manner of their interaction lies in the fact that while two masses, necessarily of the same sign, attract, two charges of the same sign repel. Thus two charges of the same sign interact in the opposite way as two masses of the same sign. The difference in the manner of the interaction and the double sign of the electric charge, can be formally brought out if we associate the factor $\sqrt{-1}$ with the electric charge.

With this idea in mind the present author has been led to study a possible modification of the theory of relativity characterized by a complex Riemannian line element in which, to the first order of approximation, the real part is associated with mass (gravitation) and the imaginary part with charge (electromagnetism). The theory thus obtained, to be presented in this paper, is distinguished by a remarkable simplicity; as will be shown the Maxwell equations and the Lorentz law of force both come out in the first approximation as a result of complex Riemannian geometry. Further the connection between charge and current densities, and electromagnetic potentials becomes at once clear.

One is led to believe that the present theory will find its place in a more comprehensive theory of matter, gravitation and electricity, as already forecast by recent developments in quantum theory, by the following argument. In the sense of the correspondence principle one must expect a unified theory of electromagnetism and gravitation for large scale phenomena which will be obtained from the complete theory in the limit

(2) The simplicity and directness of the results obtained by the method convince me that the theory here presented is probably more than just another formal divice for obtaining Maxwell's equations.

$h \to 0$. Now if in this theory Weyl's principle of gauge invariance is right, the gauge exponent must be an imaginary quantity. This necessitates the introduction of a complex line element as done in this paper.[3]

2. We postulate for the physical world a complex four-dimensional Riemannian geometry where for evident physical reasons the coordinates are real quantities. A fundamental identity is postulated to be[4]

$$(1)\quad G^{\mu}_{\nu} - \tfrac{1}{2} g^{\mu}_{\nu} G = T^{\mu}_{\nu} - \tfrac{1}{2} g^{\mu}_{\nu} T, \text{ where } T^{\mu\nu} \equiv (\varrho + i\,\sigma) \frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds},\ i \equiv \sqrt{-1},$$

and ϱ and σ are real quantities to be interpreted as the mass and charge density respectively. The dimensional factors required by the choice of units are omitted.

To find $g_{\mu\nu}$ we assume that $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$, where $\delta_{\mu\nu}$ have the Minkowski values and $h_{\mu\nu}$ are small complex quantities of the first order. Then[5]

$$(2)\quad G_{\mu\nu} = \tfrac{1}{2} g^{\sigma\varrho} \left(\frac{\partial^2 g_{\mu\nu}}{\partial x^{\sigma} \partial x^{\varrho}} + \frac{\partial^2 g_{\sigma\varrho}}{\partial x^{\mu} \partial x^{\nu}} - \frac{\partial^2 g_{\mu\sigma}}{\partial x^{\nu} \partial x^{\varrho}} - \frac{\partial^2 g_{\nu\varrho}}{\partial x^{\mu} \partial x^{\sigma}} \right)$$

which to the first approximation can be satisfied by breaking it up into two equations

$$(3)\quad G_{\mu\nu} = \tfrac{1}{2} g^{\sigma\varrho} \frac{\partial^2 g_{\mu\nu}}{\partial x^{\sigma} \partial x^{\varrho}},$$

$$(4)\quad O = g^{\sigma\varrho} \left(\frac{\partial^2 g_{\sigma\varrho}}{\partial x^{\mu} \partial x^{\nu}} - \frac{\partial^2 g_{\mu\sigma}}{\partial x^{\nu} \partial x^{\varrho}} - \frac{\partial^2 g_{\nu\varrho}}{\partial x^{\mu} \partial x^{\sigma}} \right).$$

Both will be satisfied if[6]

$$(5)\quad \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) h_{\mu\nu} = 2\, G_{\mu\nu}.$$

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(3) I owe this remark to Professor Eddington.

(4) The notation is that used in Eddington's *"The Mathematical Theory of Relativity"*.

(5) See Eddington's *l.c. p. 128.*

(6) Since here we are dealing with complex quantities it appears that we can not bring the geometry at a point to Minkowskian by real transformations of the coordinates and the validity of (5) was once doubted by Einstein. Therefore we shall give a formal proof.

From (1) we obtain

$$G^{\mu\nu} = T^{\mu\nu} + g^{\mu\nu} T,$$

which gives, neglecting $\left(\frac{dx^\nu}{ds}\right)^2$, $\left(\frac{dt}{ds} \to 1\right)$,

$$(6)\quad G^{\mu\nu} = \begin{vmatrix} 2(\rho+i\sigma) & (\rho+i\sigma)\frac{dx}{dt} & (\rho+i\sigma)\frac{dy}{dt} & (\rho+i\sigma)\frac{dz}{dt} \\ (\rho+i\sigma)\frac{dx}{dt} & -(\rho+i\sigma) & 0 & 0 \\ (\rho+i\sigma)\frac{dy}{dt} & 0 & -(\rho+i\sigma) & 0 \\ (\rho+i\sigma)\frac{dz}{dt} & 0 & 0 & -(\rho+i\sigma) \end{vmatrix}$$

$$(7)\quad G_{\mu\nu} = \begin{vmatrix} 2(\rho+i\sigma) & -(\rho+i\sigma)\frac{dx}{dt} & -(\rho+i\sigma)\frac{dy}{dt} & -(\rho+i\sigma)\frac{dz}{dt} \\ -(\rho+i\sigma)\frac{dx}{dt} & -(\rho+i\sigma) & 0 & 0 \\ -(\rho+i\sigma)\frac{dy}{dt} & 0 & -(\rho+i\sigma) & 0 \\ -(\rho+i\sigma)\frac{dz}{dt} & 0 & 0 & -(\rho+i\sigma) \end{vmatrix}$$

Evidently if (5) is satisfied (3) is satisfied. Equation (4) is equivalent to, (Eddington *l.c.* p. 129),

$$\frac{\partial}{\partial x^\alpha}\left(h^\alpha_\mu - \tfrac{1}{2}\delta^\alpha_\mu h\right) = 0.$$

Therefore our conclusion will be reached if the solutions of (5) satisfy this equation. From (5) we have

$$\Box\left(h^\alpha_\mu - \tfrac{1}{2}\delta^\alpha_\mu h\right) = 2\left(G^\alpha_\mu - \tfrac{1}{2}g^\alpha_\mu G\right) = 2\,S^\alpha_\mu,$$

where $G^\alpha_\mu - \frac{1}{2}g^\alpha_\mu G$ is written as S^σ_μ for convenience. From which we obtain

$$\frac{\partial}{\partial x_\alpha}\left(h^\alpha_\mu - \tfrac{1}{2}\delta^\alpha_\mu h\right) = \frac{1}{2\pi}\int\left\{\frac{\partial}{\partial x^\alpha}\left(S^\alpha_\mu\right)\right\}'\frac{dv}{r}.$$

But we have, *cf.* Eddington *l.c. p. 115-116*,

$$\nabla_\alpha S^\alpha_\mu = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(G^\alpha_\mu\sqrt{-g}\right) - \tfrac{1}{2}g^{\alpha\beta}\frac{\partial G_{\alpha\beta}}{\partial x^\mu}$$

$$= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(g^{\alpha\tau}g^{\sigma\rho}\sqrt{-g}\,B_{\mu\tau\sigma\rho}\right) - \tfrac{1}{2}g^{\alpha\tau}g^{\sigma\rho}\frac{\partial}{\partial x^\mu}\left(\frac{\partial^2 g_{\rho\sigma}}{\partial x^\alpha\,\partial x^\tau} - \frac{\partial^2 g_{\rho\tau}}{\partial x^\nu\,\partial x^\sigma}\right)$$

$$= g^{\alpha\tau}g^{\sigma\rho}\frac{\partial}{\partial x_\alpha}B_{\mu\tau\sigma\rho} - \tfrac{1}{2}g^{\alpha\tau}g^{\sigma\delta}\frac{\partial}{\partial x^\mu}\left(\frac{\partial^2 g_{\rho\sigma}}{\partial x^\alpha\,\partial x^\tau} - \frac{\partial^2 g_{\rho\sigma}}{\partial x^\nu\,\partial x^\sigma}\right).$$

The differential equations (5) can be readily solved and they give,

$$(8)\quad g_{\mu\nu}\equiv\begin{vmatrix} 1-2u-2i\Phi & a_x+iA_x & ay+iA_y & a_z+iA_z \\ a_x+iA_x & -(1-\mu-i\Phi) & 0 & 0 \\ a_y+iA_y & 0 & -(1-\mu-i\Phi) & 0 \\ a_z+iA_z & 0 & 0 & -(1-\mu-i\Phi) \end{vmatrix},$$

where

$$\mu\equiv\iiint\left\{\frac{\rho d\mathbf{v}}{r}\right\}'_{t-\frac{r}{c}},\quad a_x\equiv\iiint\left\{\frac{\rho\frac{dx}{dt}d\mathbf{v}}{r}\right\}'_{t-\frac{r}{c}},$$

$$a_y\equiv\iiint\left\{\frac{\rho\frac{dy}{dt}d\mathbf{v}}{r}\right\}'_{t-\frac{r}{c}},\ \text{etc.}$$

(9)

$$\Phi\equiv\iiint\left\{\frac{\sigma d\mathbf{v}}{r}\right\}_{t-\frac{r}{c}},\quad A_x\equiv\iiint\left\{\frac{\sigma\frac{dx}{dt}d\mathbf{v}}{r}\right\}'_{t-\frac{r}{c}},$$

$$A_y\equiv\iiint\left\{\frac{\sigma\frac{dx}{dt}d\mathbf{v}}{r}\right\}_{t-\frac{r}{c}},\ \text{etc.}$$

and constant factors $-\frac{1}{4\pi}$ have been omitted by the proper choice of the units. Thus to the desired order of approximation the Riemannian line element is

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For $B_{\mu\tau\sigma\rho}$ is at least of the first order and $\frac{\partial g_{\alpha\beta}}{\partial x^\nu}$ is also of first order, hence $g^{\nu\tau}$'s are taken outside the sign of differentiation. The last expression is seen *l.c. p. 116*, to cancel each other. Furthermore

$$\nabla_\alpha S^\alpha_\mu=\frac{\partial}{\partial x^\alpha}S^\alpha_\mu+\left\{\begin{matrix}\alpha\beta\\\beta\end{matrix}\right\}S^\alpha_\mu-\left\{\begin{matrix}\beta\mu\\\alpha\end{matrix}\right\}S^\beta_\alpha,$$

and the last two terms are evidently of at least second order. Thus

$$\frac{\partial}{\partial x^\alpha}\left(h^\alpha_\mu-\tfrac{1}{2}\delta^\alpha_\mu h\right)=0$$

to the degree of approximation desired.

(10)

$$ds^2=\left\{(1-2\mu)\ dt^2+a_x dxdt+a_y dydt+a_z dzdt-(1-\mu)(dx^2+dy^2+dz^2)\right\}$$
$$+\sqrt{-1}\left\{-2\Phi dt^2+A_x dx\ dt+A_y dy\ dt+A_z dzdt+\Phi(dx^2+dy^2+dz^2)\right\}.$$

Thus in the first approximation the gravitational and electromagnetic fields are kept separate in the Riemannian line element, which is in agreement with experimental facts. The former occurs in the real part of the metric which contains the mass density through the potentials μ, a_x, a_y, a_z and the latter in the imaginary part which contains the charge density through the potentials Φ, A_x, A_y, A_z. It is seen that the expression for ds^2 does not reduce to the Minkowskian line element unless both the charge and the mass density vanish. However at any point the real part of the metric can be reduced to the Minkowskian ds^2 without *thereby* destroying the electromagnetic field.

3. For equation of motion we need another postulate. Taking Eulerian hydrodynamical equation as a guide we assume therefore that the real part of the divergence of $T^{\mu\nu}$

(11) $$R\ (\nabla_\nu T^{\mu\nu})=0$$

as our chosen equation of motion. That the above equations are tensor equations can be easily seen.

This equation, when integrated through a small four-dimensional volume following the exact procedure as given in Eddington's *l.c.* § 56 p. 126, gives the equation of motion of a particle as

(12) $$R\left\{(m+ie)\frac{d^2x^\nu}{ds^2}+(m+ie)\left\{\begin{matrix}\alpha\beta\\ \nu\end{matrix}\right\}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right\}=0.$$

To evaluate this equation we first calculate the Christoffels symbols and obtain,

(13) $$\begin{bmatrix}11\\ 1\end{bmatrix}=-\frac{\partial\mu}{\partial t}-i\frac{\partial\Phi}{\partial t},$$

$$\begin{bmatrix}11\\ 2\end{bmatrix}=\frac{\partial a_x}{\partial t}+\frac{\partial x}{\partial\mu}+i\left(\frac{\partial A_x}{\partial t}+\frac{\partial\Phi}{\partial x}\right),$$

$$\begin{bmatrix}11\\3\end{bmatrix}=\frac{\partial\alpha_y}{\partial t}+\frac{\partial\mu}{\partial y}+i\left(\frac{\partial A_y}{\partial t}+\frac{\partial\Phi}{\partial y}\right),$$

$$\begin{bmatrix}11\\4\end{bmatrix}=\frac{\partial\alpha_z}{\partial t}+\frac{\partial\mu}{\partial z}+i\left(\frac{\partial A_z}{\partial t}+\frac{\partial\Phi}{\partial z}\right),$$

$$\begin{bmatrix}12\\3\end{bmatrix}=\frac{1}{2}\left(\frac{\partial\alpha_y}{\partial x}-\frac{\partial\alpha_x}{\partial y}\right)+\frac{1}{2}i\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right),$$

$$\begin{bmatrix}13\\2\end{bmatrix}=\frac{1}{2}\left(\frac{\partial\alpha_x}{\partial y}-\frac{\partial\alpha_y}{\partial x}\right)+\frac{1}{2}i\left(\frac{\partial A_x}{\partial y}-\frac{\partial A_y}{\partial x}\right).$$

$$\begin{bmatrix}14\\2\end{bmatrix}=\frac{1}{2}\left(\frac{\partial\alpha_x}{\partial z}-\frac{\partial\alpha_z}{\partial x}\right)+\frac{1}{2}i\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)$$

$$\begin{bmatrix}13\\4\end{bmatrix}=\frac{1}{2}\left(\frac{\partial\alpha_z}{\partial y}-\frac{\partial\alpha_y}{\partial z}\right)+\frac{1}{2}i\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right),\ etc.$$

$$\begin{bmatrix}12\\2\end{bmatrix}=\begin{bmatrix}13\\3\end{bmatrix}=\begin{bmatrix}14\\4\end{bmatrix}=\frac{1}{2}\frac{\partial\mu}{\partial t}+\frac{1}{2}i\frac{\partial\Phi}{\partial t}\ etc.$$

$$\begin{Bmatrix}\mu\nu\\\alpha\end{Bmatrix}=\begin{bmatrix}\mu\nu\\\alpha\end{bmatrix}\ if\ \alpha=1,\quad \begin{Bmatrix}\mu\nu\\\alpha\end{Bmatrix}=-\begin{bmatrix}\mu\nu\\\alpha\end{bmatrix},\ if\ \alpha=2,3,4.$$

Consider the x-component of the equation, $\nu=2$,

$$(14)\quad R\left\{(m+ie)\frac{d^2x}{dt^2}+(m+ie)\left(\begin{Bmatrix}11\\2\end{Bmatrix}+2\begin{Bmatrix}12\\2\end{Bmatrix}\frac{dx}{dt}+2\begin{Bmatrix}13\\2\end{Bmatrix}\frac{dy}{dt}\right.\right.$$
$$\left.\left.+2\begin{Bmatrix}14\\2\end{Bmatrix}\frac{dz}{dt}\right)\right\}=m\frac{d^2x}{dt^2}-m\left[\frac{\partial\alpha_x}{\partial t}+\frac{\partial\mu}{\partial x}+\dot{x}\frac{\partial\mu}{\partial t}+\dot{y}\left(\frac{\partial\alpha_x}{\partial y}-\frac{\partial\alpha_y}{\partial x}\right)\right.$$
$$\left.+\dot{z}\left(\frac{\partial\alpha_x}{\partial z}-\frac{\partial\alpha_z}{\partial x}\right)\right]+e\left[\frac{\partial A_x}{\partial t}+\frac{\partial\Phi}{\partial x}+\dot{x}\frac{\partial\Phi}{\partial t}+\dot{y}\left(\frac{\partial A_x}{\partial y}-\frac{\partial A_y}{\partial x}\right)\right.$$
$$\left.+\dot{z}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\right]=0$$

If we introduce a vector $\vec{H}$ defined by

$$(15)\qquad (H_x, H_y, H_z)=curl\ (A_x, A_y, A_z),$$

and a vector $\vec{E}$ defined by

$$(16)\qquad (E_x, E_y, E_z)=-\frac{\partial}{\partial t}(A_x, A_y, A_z)-grad\ \Phi,$$

and the cross product $[\vec{V}\vec{H}]$ then we have the equation of motion of a particle as

$$(17) \quad m\frac{d^2x}{dt^2} - m\left\{\dot{a}_x + \frac{\partial\mu}{\partial x} + \dot{x}\,\frac{\partial\mu}{\partial t} - [\vec{V}\vec{h}\,]_x\right\} - e\left\{E_x + [\vec{V}\vec{H}]_x - \dot{x}\,\frac{\partial\Phi}{\partial t}\right\} = 0,$$

or

$$(18) \quad \left\{m\frac{d^2x}{dt^2} = m\,\frac{\partial\mu}{\partial x} + eE_x + e\,[\vec{V}\vec{H}\,]_x\right\} + m\,(\dot{a}_x - [\vec{V}\vec{h}]_x) + m\dot{x}\,\frac{\partial\mu}{\partial t} - e\dot{x}\,\frac{\partial\Phi}{\partial t}$$

The first terms in brackets are the classical laws of motion of a charged particle under the influence of gravitational and electromagnetic forces; the other terms are in the nature of small corrections.

It is seen from (15) and (16) that the vectors $\vec{E}$ and $\vec{H}$ are derived from the retarded potentials in the same way as the electric and magnetic vectors of Maxwell's theory.

4. We conclude by examining the rôle of the Lorentz transformation in the present theory. To begin with we note that in the present theory Maxwell's equations are not given *a priori* but deduced from fundamental geometrical postulates in the same way as Newton's law of gravitation was deduced by Einstein in 1916, to the same order of approximation. If now there are two observers in motion with respect to each other we might inquire how far the approximate methods used in this paper for the derivation of Maxwell's equations and Lorentz law of force are valid. From (5) it is seen that the present treatment is legitimate only if the two systems of reference are connected with each other which keeps $\square$ invariant, i.e. the Lorentz transformation.

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