

# 非线性波动方程的孤波解<sup>\*</sup>

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用平衡法并结合吴消元法得到了一类较广泛非线性波动方程  $u_{tt} = a_1 u_{xx} + a_2 u_t + a_3 u + a_4 u^3 = 0$  的若干孤波解公式, 从而物理学上许多著名的方程, 如  $\phi^4$  方程、Klein-Gordon 方程、Landau-Ginzburg-Higgs 方程、非线性电报方程等都可作为该方程的特殊情形得到相应的孤波解.

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## 1 引 言

寻找非线性数学物理方程的孤波解长期以来一直受到数学家和物理学家的关注. 目前已发现越来越多的具有重要物理意义的非线性演化方程, 如 sin-Gordon 方程、KdV 方程、Schrödinger 方程等<sup>[1,6,8,9]</sup> 均具有孤波解. 孤子正在流体物理学、固体物理学、基本粒子物理学、等离子体物理学、凝聚态物理学等许多领域找到了应用, 并对一些过去难以解释的现象作了说明, 如有人找到了旋转大气波的孤子解用来解释木星的红斑和其他特征. 寻找孤子解的方法多种多样, 如借助于函数变换、Backlund 变换、Hirota 变换、Darboux 方法等<sup>[1,6,8,9]</sup>.

本文用最近引入的齐次平衡法<sup>[7]</sup>的思想, 求如下一类较广泛非线性波动方程

$$u_{tt} - a_1 u_{xx} + a_2 u_t + a_3 u + a_4 u^3 = 0 \quad (1)$$

的孤波解, 其中  $a_1, a_2, a_3, a_4$  为常数. 求解(1)式很有意义, 因该方程可简化为物理学上许多著名的方程, 如 sin-Gordon 方程、sinh-Gordon 方程的近似, 以及  $\phi^4$  方程、Klein-Gordon 方程、Landau-Ginzburg-Higgs 方程、非线性电报方程等<sup>[1-9]</sup>.

## 2 非线性波动方程(1)的孤波解

设方程(1)的解具有如下形式:

$$u \equiv f'(\varphi) \varphi_x + \alpha, \quad \varphi_t \equiv \beta \varphi_x, \quad (2)$$

其中  $f, \varphi$  为待定函数,  $\alpha, \beta$  为待定常数. 由(2)式经计算, 可得

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$$\begin{aligned}
 u_{tt} &= \beta^2 f''' \varphi_x^3 + 3\beta^2 f'' \varphi_x \varphi_{xx} + \beta^2 f' \varphi_{xxx}, \\
 -a_1 u_{xx} &= -a_1 f''' \varphi_x^3 - 3a_1 f'' \varphi_x \varphi_{xx} - a_1 f' \varphi_{xxx}, \\
 a_2 u_t &= a_2 \beta f'' \varphi_x^2 + a_2 \beta f' \varphi_{xx}, \\
 a_3 u &= a_3 f' \varphi_x + a_3 \alpha, \\
 a_4 u^3 &= a_4 f^3 \varphi_x^3 + 3a_4 \alpha f^2 \varphi_x^2 + 3a_4 \alpha^2 f' \varphi_x + a_4 \alpha^3.
 \end{aligned}$$

将以上各式代入(1)式,得

$$\begin{aligned}
 &u_{tt} - a_1 u_{xx} + a_2 u_t + a_3 u + a_4 u^3 \\
 &= [(\beta^2 - a_1) f''' + a_4 f^3] \varphi_x^3 + (3\beta^2 f'' \varphi_x \varphi_{xx} \\
 &\quad - 3a_1 f'' \varphi_x \varphi_{xx} + a_2 \beta f'' \varphi_x^2 + 3a_4 \alpha f^2 \varphi_x^2) \\
 &\quad + [(\beta^2 - a_1) \varphi_{xxx} + a_2 \beta \varphi_{xx} + 3a_4 \alpha^2 \varphi_x + a_3 \varphi_x] f' \\
 &\quad + \alpha(a_3 + a_4 \alpha^2) = 0.
 \end{aligned} \tag{3}$$

令  $\varphi_x^3$  的系数为零,即

$$(\beta^2 - a_1) f''' + a_4 f^3 = 0. \tag{4}$$

当  $a_4(a_1 - \beta^2) > 0$ , 解之得

$$f = \pm \sqrt{\frac{2(a_1 - \beta^2)}{a_4}} \ln \varphi. \tag{5}$$

由此可得

$$f'^2 = \mp \sqrt{\frac{2(a_1 - \beta^2)}{a_4}} f''. \tag{6}$$

将(6)式代入(3)式,并利用(4)式,得

$$\begin{aligned}
 &u_{tt} - a_1 u_{xx} + a_2 u_t + a_3 u + a_4 u^3 \\
 &= \left[ 3(\beta^2 - a_1) \varphi_x \varphi_{xx} + a_2 \beta \varphi_x^2 \mp 3\alpha a_4 \sqrt{\frac{2(a_1 - \beta^2)}{a_4}} \varphi_x^2 \right] f'' \\
 &\quad + [(\beta^2 - a_1) \varphi_{xxx} + a_2 \beta \varphi_{xx} + a_3 \varphi_x + 3a_4 \alpha^2 \varphi_x] f' + \alpha(a_3 + a_4 \alpha^2) = 0.
 \end{aligned}$$

再令  $f''$ ,  $f'$  的系数及常数项为零,可得如下方程组:

$$3(\beta^2 - a_1) \varphi_x \varphi_{xx} + \left( a_2 \beta \mp 3\alpha a_4 \sqrt{\frac{2(a_1 - \beta^2)}{a_4}} \right) \varphi_x^2 = 0, \tag{7}$$

$$(\beta^2 - a_1) \varphi_{xxx} + a_2 \beta \varphi_{xx} + (a_3 + 3a_4 \alpha^2) \varphi_x = 0, \tag{8}$$

$$\alpha(a_3 + a_4 \alpha^2) = 0.$$

设  $\varphi(x, t)$  具有如下形式的解:

$$\varphi(x, t) = 1 + e^{l(x + \beta t + c)}, \tag{9}$$

其中  $l$  为待定常数,  $c$  为任意常数,则  $\varphi(x, t)$  必须满足(7)和(8)式,只要

$$3(\beta^2 - a_1) l + a_2 \beta \mp 3a_4 \alpha \sqrt{\frac{2(a_1 - \beta^2)}{a_4}} = 0, \tag{10}$$

$$(\beta^2 - a_1) l^2 + a_2 \beta l + a_3 + 3a_4 \alpha^2 = 0, \tag{11}$$

$$\alpha(a_3 + a_4 \alpha^2) = 0. \quad (12)$$

利用吴消元法解上述多项式方程组, 可得

i) 若  $a_2 = 0$ , (10)–(12) 式有解

$$\alpha = \pm \sqrt{\frac{-a_3}{a_4}}, \quad l = \mp \sqrt{\frac{-2a_3}{a_1 - \beta^2}}, \quad a_4(a_1 - \beta^2) > 0, \quad a_3 a_4 < 0.$$

$$\alpha = \pm i \sqrt{\frac{a_3}{a_4}}, \quad l = \mp i \sqrt{\frac{2a_3}{a_1 - \beta^2}}, \quad a_4(a_1 - \beta^2) > 0, \quad a_3 a_4 > 0.$$

ii) 若  $a_2 \neq 0$ , (10)–(12) 式有解

$$\alpha = 0, \quad \beta = \pm 3 \sqrt{\frac{a_1 a_3}{9a_3 - 2a_2^2}}, \quad l = \mp \frac{a_3}{2a_2} \sqrt{\frac{9a_3 - 2a_2^2}{a_1 a_3}},$$

$$a_1 a_3 (9a_3 - 2a_2^2) > 0, \quad a_3 a_4 < 0.$$

$$\alpha = \sqrt{\frac{-a_3}{a_4}}, \quad \beta = \pm 3 \operatorname{sgn}\left(\frac{a_4}{a_2}\right) \sqrt{\frac{a_1 a_3}{9a_3 - 2a_2^2}},$$

$$l = \pm \operatorname{sgn}\left(\frac{a_4}{a_2}\right) \frac{a_3}{2a_2} \sqrt{\frac{9a_3 - 2a_2^2}{a_1 a_3}}, \quad a_1 a_3 (9a_3 - 2a_2^2) > 0, \quad a_3 a_4 < 0.$$

$$\alpha = -\sqrt{\frac{-a_3}{a_4}}, \quad \beta = \mp 3 \operatorname{sgn}\left(\frac{a_4}{a_2}\right) \sqrt{\frac{a_1 a_3}{9a_3 - 2a_2^2}},$$

$$l = \mp \operatorname{sgn}\left(\frac{a_4}{a_2}\right) \frac{a_3}{2a_2} \sqrt{\frac{9a_3 - 2a_2^2}{a_1 a_3}}, \quad a_1 a_3 (9a_3 - 2a_2^2) > 0, \quad a_3 a_4 < 0.$$

于是由(2), (5), (9) 及 i), ii), 得如下定理:

定理 i) 当  $a_2 = 0$ , 方程(1)有扭状孤波解

$$u = \pm \sqrt{\frac{-a_3}{a_4}} \tanh \sqrt{\frac{-a_3}{2(a_1 - \beta^2)}} (x + \beta t + c), \quad a_4(a_1 - \beta^2) > 0, \quad a_3 a_4 < 0, \quad (13)$$

$$u = \pm \sqrt{\frac{a_3}{a_4}} \tan \sqrt{\frac{a_3}{2(a_1 - \beta^2)}} (x + \beta t + c), \quad a_4(a_1 - \beta^2) > 0, \quad a_3 a_4 > 0. \quad (14)$$

ii) 当  $a_2 \neq 0$ , 注意到  $u(x, t)$ ,  $u(-x, t)$  同为方程(1)的解, 则方程(1)有扭状孤波解

$$u = \pm \frac{|a_2|}{2a_2} a_3 \sqrt{-\frac{1}{a_3 a_4}} \tanh \frac{a_3}{4a_2} \sqrt{\frac{9a_3 - 2a_2^2}{a_1 a_3}} \left( x + 3 \sqrt{\frac{a_1 a_3}{9a_3 - 2a_2^2}} t + c \right) \\ \pm \frac{|a_2|}{2a_2} a_3 \sqrt{-\frac{1}{a_3 a_4}}, \quad a_3 a_4 < 0, \quad a_1 a_3 (9a_3 - 2a_2^2) > 0. \quad (15)$$

### 3 应用举例

1. 令  $a_1 = 1$ ,  $a_2 = 0$ ,  $a_3 = 1$ ,  $a_4 = -\frac{1}{6}$ , 由(13)式可得 sin-Gordon 方程的近似方程<sup>[1,6]</sup>

$$u_{tt} - u_{xx} + u - \frac{1}{6} u^3 = 0 \text{ 的两个孤波解}$$

$$u = \pm \sqrt{6} \tanh \frac{1}{\sqrt{2(\beta^2 - 1)}} (x + \beta t + c) \quad \beta^2 > 1.$$

2. 令  $a_1 = 1, a_2 = 0, a_3 = 1, a_4 = \frac{1}{6}$ , 由(14)式可得 sin-Gordon 方程的近似方程<sup>[1,6]</sup>

$$u_{tt} - u_{xx} + u + \frac{1}{6} u^3 = 0 \text{ 的两个孤波解}$$

$$u = \pm \sqrt{6} \tan \frac{1}{\sqrt{2(1 - \beta^2)}} (x + \beta t + c) \quad \beta^2 < 1.$$

3. 令  $a_1 = 1, a_2 = 0, a_3 = m^2, a_4 = \lambda < 0$ , 由(13)式可得 Klein-Gordon 方程<sup>[1,6,8]</sup>  $u_{tt} - u_{xx} + m^2 u + \lambda u^3 = 0$  的两个孤波解

$$u = \pm \frac{m}{\sqrt{-\lambda}} \tanh \frac{m}{\sqrt{2(\beta^2 - 1)}} (x + \beta t + c) \quad \beta^2 > 1.$$

4. 令  $a_1 = 1, a_2 = 0, a_3 = -m^2, a_4 = g^2$ , 由(13)式可得 Landou-Ginburg-Higgs 方程<sup>[3]</sup>  $u_{tt} - u_{xx} - m^2 u + g^2 u^3 = 0$  的两个孤波解

$$u = \pm \frac{m}{g} \tanh \frac{m}{\sqrt{2(1 - \beta^2)}} (x + \beta t + c) \quad \beta^2 < 1.$$

5. 令  $a_1 = a_2 = 0$ , 而  $u$  仅是  $t$  的函数, 由(13)和(14)式, 可得 Duffing 方程<sup>[2,6]</sup>  $u_{tt} + a_3 u + a_4 u^3 = 0$  的两个孤波解

$$u = \pm \sqrt{\frac{-a_3}{a_4}} \tanh \sqrt{\frac{a_3}{2}} (t + c) \quad a_3 > 0, \quad a_4 < 0,$$

$$u = \pm \sqrt{\frac{a_3}{a_4}} \tan \sqrt{\frac{-a_3}{2}} (t + c) \quad a_3 < 0, \quad a_4 < 0.$$

6. 由(15)式可得非线性电报方程<sup>[4,5]</sup>  $u_{tt} - a_1 u_{xx} + a_2 u_t + a_3 u + a_4 u^3 = 0$  的两个孤波解

$$u = \pm \frac{|a_2|}{a_2} a_3 \sqrt{-\frac{1}{a_3 a_4}} \tanh \frac{a_3}{4 a_2} \sqrt{\frac{9 a_3 - 2 a_2^2}{a_1 a_3}} \left( x + 3 \sqrt{\frac{a_1 a_3}{9 a_3 - 2 a_2^2}} t + c \right) \\ \pm \frac{|a_2|}{2 a_2} a_3 \sqrt{-\frac{1}{a_3 a_4}} \quad a_3 a_4 < 0, \quad a_1 a_3 (9 a_3 - 2 a_2^2) > 0.$$

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# THE SOLITARY WAVE SOLUTIONS FOR A CLASS OF NONLINEAR WAVE EQUATIONS

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## ABSTRACT

Some solitary wave solutions for a class of nonlinear wave equations  $u_{tt} - a_1 u_{xx} + a_2 u_t + a_3 u + a_4 u^3 = 0$  are obtained by homogeneous balance method and Wu-elimination method. Many well-known equations in physics such as  $\phi^4$  equation, Klein-Gordon equation, Duffing equation and telegraph equation are special cases of the wave equation presented in this paper.

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