VELOCITY AND TEMPERATURE DISTRIBUTIONS IN TURBULENT WAKES BEHIND AN INFINITE CYLINDER AND A BODY OF REVOLUTION

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ABSTRACT

Chou's theory of turbulence is applied to investigate the velocity and temperature distributions in turbulent wakes behind an infinite cylinder and a body of revolution. The results of the theory for the mean squares of velocity fluctuations across the wake, and for the distributions of mean velocity and temperature behind the wakes are found to agree well with the experiments of Fage, of Fage and Falkner, and of Hall and Hislop.

1. *Introduction.*

Recently P. Y. Chou[1] has developed a new theory of turbulence based upon a set of differential equations of turbulent fluctuation which, as a matter of fact, supplement Reynolds' original equations of mean motion. To solve these equations of fluctuctions he constructs the double and triple correlation functions and the differential equations satisfied by them along the lines followed by Taylor and von Kármán in their statistical theory of isotropic turbulence. He has applied the general theory to solve the problem of pressure flow within a channel and C. C. Lin[2] has found the solution of pressure flow in a circular pipe. Both of them have neglected the action of viscosity and obtained results which are in good agreement with the experimental data. It is found that according to Chou's theory the distributions of mean velocity and of mean squares of turbulent velocity fluctuations are correlated and hence the measurements of them should be performed in a single experiment at the same Reynolds number in order to be compared accurately with the theoreticel predictions.

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1. P. Y. Chou, *Chinese J. Phys.* 4 (1940), 1-33
2. C. C. Lin, "Pressure Flow of a Turbulent Fluid through a Circular Pipe": (unpublished)

The present paper deals with the solution of the velocity and temperature distributions in the wakes behind an infinite cylinder and a body of revolution according to the same theory. It will be shown below that in this problem only the equations of mean motion and the equations of double correleation are used while the third order velocity correlations can be assumed either to be constants across the wake (if they are even functions) or to be proportional to the distance perpendicular to the plane or to the axis of symmetry (if they are odd functions of this distance), so that the differential equations satisfied by these correlations can be left out. As we shall see soon, the theoretical predictions of the mean squares of velocity fluctuations across the wake behind an infinite cylinder and the axially symmetrical body all agree very well with the experimental measurements of Fage, of Fage and Falkner and of Hall and Hislop respectively.

For the sake of generality, we shall write our equations in curvilinear coordinates using the language of tensor analysis. For details the reader can consult Chou's original paper. Then the equations of steady mean flow can be written as:

$$(1.1) \qquad U^j U_{i,j} = -\frac{1}{\rho} p_{,i} + \frac{1}{\rho} \tau_{i,j}^{\;j} + \nu \nabla^2 U_i, \qquad U^i_{,i} = 0$$

and the equations of steady double correlations are given by

$$-\frac{1}{\rho}\left(U_{i,j}\tau^j_k + U_{k,j}\tau^j_i\right) - \frac{1}{\rho} U^j \tau_{ik,j} + \left(\overline{w^j w_i w_k}\right)_{,j}$$

$$(1.2) \qquad = -\frac{1}{\rho}\left(\overline{\varpi_{,i} w_k + \varpi_{,k} w_i}\right) - \frac{\nu}{\rho} \nabla^2 \tau_{ik}$$

$$-2\nu g^{mn} \overline{w_{i,m} w_{k,n}}$$

where U_j and w_j denote the mean velocity and the velocity fluctuation, p and ϖ the mean pressure and pressure fluctuation respectively and $\tau^{ij} = -\rho \overline{w_i w_j}$, the apparent stress. The bar over the quantities means the average values of the corresponding quantities over time and the index behind a comma denotes the covariant differentiation of the quantities along the direction denoted by that index. Equations (1.1) and (1.2) constitute the fundamental equations for the present treatment. In the following sections we shall apply the theory first to the two-dimensional wake behind an infinite cylinder and then to the three-dimensional axially symmetrical wake behind a body of revolution in succession.

2. The two-dimensional wake behind an infinite cylinder: Velocity distribution.

Before we start with the solution of the problem, let us examine the equations (1.1) and (1.2). In (1.1) the term $\nu\nabla^2 U_i$ unles very near to the solid obstacle, is negligible when compared with the other terms, and in (1.2) the terms $\nu\nabla^2\tau_{ik}$ have also been shown by von Kármán[3] to be much smaller than the terms following them, so in the following analysis we shall drop these terms completely. Furthermore we shall also neglect the term $\frac{1}{\rho}\nabla(\overline{w_i w_k}+\overline{w_k w_i})$ in (1.2). This is what has been done by Chou and Lin in the cases of turbulent flow through the channel and the pipe. And lastly, according to von Kármán we may replace the term $2\nu g^{mn}\overline{w_{i,m}w_{k,n}}$ by $2\nu k_{(ik)}\overline{w_i w_k}/\lambda^2$ where λ is Taylor's scale of micro-turbulence as defined in the theory of isotropic turbulence and $k_{(ik)}$ are constants which are equal to 5 when $i=k$ and indeterminate when $i\neq k$ in the case of isotropic turbulence. The parenthesis about the indices i, k are introduced to indicate that the summation convention is suspended.

We shall now consider the wake behind an infinite cylinder. Let us use rectangular coordinates and choose the x-axis along the direction of the undisturbed flow outside the wake and the z-axis along the axis of the cylinder. Then the equations in (1.1) and (1.2) which are not identically zero with the above modifications can be written respectively as:

$$
(2.1)\quad \begin{cases} U\dfrac{\partial U}{\partial x}+V\dfrac{\partial U}{\partial y}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+\dfrac{1}{\rho}\left(\dfrac{\partial\tau_{xx}}{\partial x}+\dfrac{\partial\tau_{xy}}{\partial y}\right) \\ U\dfrac{\partial V}{\partial x}+V\dfrac{\partial V}{\partial y}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}+\dfrac{1}{\rho}\left(\dfrac{\partial\tau_{yx}}{\partial x}+\dfrac{\partial\tau_{yy}}{\partial y}\right) \end{cases}
$$

and

$$
\begin{cases} -\dfrac{2}{\rho}\left(\tau_{xx}\dfrac{\partial U}{\partial x}+\tau_{xy}\dfrac{\partial U}{\partial y}\right)-\dfrac{1}{\rho}U\dfrac{\partial\tau_{xx}}{\partial x}-\dfrac{1}{\rho}V\dfrac{\partial\tau_{xx}}{\partial y} \\ \qquad +\dfrac{\partial}{\partial x}\overline{w_x{}^3}+\dfrac{\partial}{\partial y}\overline{w_x{}^2 w_y}=\dfrac{2\nu k_{(11)}}{\rho\lambda^2}\tau_{xx}, \end{cases}
$$

3. Th. von Kármán, *Jour. of Aero. Sci. 4* (1937), 135

$$
(2.2)\quad \begin{cases}
-\dfrac{1}{\rho}\left(\tau_{xx}\dfrac{\partial V}{\partial x}+\tau_{yy}\dfrac{\partial U}{\partial y}+\tau_{xy}\left(\dfrac{\partial U}{\partial x}+\dfrac{\partial V}{\partial y}\right)\right) \\
-\dfrac{1}{\rho}U\dfrac{\partial \tau_{xy}}{\partial x}-\dfrac{1}{\rho}V\dfrac{\partial \tau_{xy}}{\partial y}+\dfrac{\partial}{\partial x}\overline{w_x{}^2 w_y} \\
+\dfrac{\partial}{\partial y}\overline{w_x w_y{}^2}=\dfrac{2\nu k_{(12)}}{\rho\lambda^2}\tau_{xy}, \\
-\dfrac{2}{\rho}\left(\tau_{xy}\dfrac{\partial V}{\partial x}+\tau_{yy}\dfrac{\partial V}{\partial y}\right)-\dfrac{1}{\rho}U\dfrac{\partial \tau_{yy}}{\partial x}-\dfrac{1}{\rho}V\dfrac{\partial \tau_{yy}}{\partial y} \\
+\dfrac{\partial}{\partial x}\overline{w_x w_y{}^2}+\dfrac{\partial}{\partial y}\overline{w_y{}^3}=\dfrac{2\nu k_{(22)}}{\rho\lambda^2}\tau_{yy}, \\
-\dfrac{1}{\rho}\left(U\dfrac{\partial \tau_{zz}}{\partial x}+V\dfrac{\partial \tau_{zz}}{\partial y}\right)+\dfrac{\partial}{\partial x}\overline{w_x w_z{}^2}+\dfrac{\partial}{\partial y}\overline{w_y w_z{}^2} \\
=\dfrac{2\nu k_{(33)}}{\rho\lambda^2}\tau_{zz}
\end{cases}
$$

where U and V are the x and y components of the mean velocity respectively.

Since (1) it has been found experimentally that there is dynamical similarity in the different cross-sections of the wake when the distance behind the obstacle is large, (2) the boundary of the wake behind an infinite cylinder is found to be a cylindrical surface $y^2=$const. x, with its vertex line coinciding with the axis of the cylinder and (3) the dragging force, given by the integral, $\rho\, U_0 \int_{-\infty}^{\infty}(U_0-U)dy$ (where U_0 is the undisturbed velocity outside the wake), must be independent of the position of the plane of integration, then U must be of the following form:

$$
(2.3)\qquad U=U_0\left\{1-\frac{1}{\sqrt{x}}F_1(\eta)\right\},\quad \eta=y\Big/\sqrt{x}\,.
$$

From the equation of continuity for an incompressible fluid,

$$
(2.4)\qquad \frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0
$$

we can immediately write

$$
(2.5)\qquad V=\frac{U_0}{x}F_2(\eta)
$$

so that $F_1(\eta)$ and $F_2(\eta)$ are two unknown functions to be determined presently.

It is seen that both U_0-U and V decrease as x increases. If we consider $\varepsilon=1/\sqrt{x}$ as a small quantity of the first order, then

$$
(2.6)\quad \left\{
\begin{array}{llll}
U_0-U \sim \varepsilon \;, & V \sim \varepsilon^2 \;, & \dfrac{\partial U}{\partial y} \sim \varepsilon^2 \;, & \dfrac{\partial U}{\partial x} \sim \varepsilon^3 \;, \\[2ex]
\dfrac{\partial V}{\partial y} \sim \varepsilon^3 \;, & \dfrac{\partial V}{\partial x} \sim \varepsilon^4 \;, & &
\end{array}
\right.
$$

as can be verified by actual differentiation. Now let the velocity fluctuation w_i be of the order ε^s. Then we have

$$
(2.7)\qquad \tau_{xy},\ \tau_{yy},\ \text{etc.} \sim \varepsilon^{2s};\ \overline{w_x^3},\ \overline{w_x w_y^2},\ \text{etc.} \sim \varepsilon^{3s}
$$

If we neglect all small quantities above the third order in the first equation of (2.1) when x is large, then we obtain

$$
\frac{\partial \tau_{xy}}{\partial y} = -\rho U_0 \frac{\partial U}{\partial x} \sim \varepsilon^3
$$

Hence we have $s=1$.

Using the above value of s, we may put

$$
(2.8)\quad \left\{
\begin{array}{ll}
-\dfrac{1}{\rho}\,p = \dfrac{U_0^2}{x} f_0(\eta), & \overline{w_x^2} = \dfrac{U_0^2}{x} f_1(\eta), \\[2ex]
\overline{w_x w_y} = \dfrac{U_0^2}{x} f_2(\eta), & \overline{w_y^2} = \dfrac{U_0^2}{x} f_3(\eta), \\[2ex]
\overline{w_z^2} = \dfrac{U_0^2}{x} f_4(\eta), & \overline{w_x^3} = \dfrac{U_0^3}{x^{3/2}} h_0(\eta), \\[2ex]
\overline{w_x^2 w_y} = \dfrac{U_0^3}{x^{3/2}} h_1(\eta), & \overline{w_x w_y^2} = \dfrac{U_0^3}{x^{3/2}} h_2(\eta), \\[2ex]
\overline{w_y^3} = \dfrac{U_0^3}{x^{3/2}} h_3(\eta), & \overline{w_x w_z^2} = \dfrac{U_0^3}{x^{3/2}} h_4(\eta), \\[2ex]
\overline{w_y w_z^2} = \dfrac{U_0^3}{x^{3/2}} h_5(\eta) &
\end{array}
\right.
$$

Substituting (2.3), (2.5) and (2.8) into the equations (2.1), (2.4) and (2.2) and neglecting all terms of high orders except the lowest in each equation, then we obtain the two equations of mean motion after being integrated once with respect to η, simplified into the form,

$$(2.9)\qquad \eta F_1 = -2f_2, \qquad f_0 = f_3 + \text{const.};$$

the equation of continuity for mean motion as

$$(2.10)\qquad F_2 = -\tfrac{1}{2}\eta F_1 \quad ;$$

and the equations of double correlations to be given by

$$(2.11)\qquad \begin{cases} 2f_2 F'_1 + f_0 + \frac{1}{2}\eta f'_1 - h'_1 = k(11)\, l f_1 \\ f_3 F'_1 + f_2 + \frac{1}{2}\eta f'_2 - h'_2 = k(12)\, l f_2 \\ f_3 + \frac{1}{2}\eta f'_3 - h'_3 = k(22)\, l f_3 \\ f_4 + \frac{1}{2}\eta f'_4 - h'_5 = k(33)\, l f_4 \end{cases}$$

where the dash denotes differentiation with respect to η, and $l = 2\nu x/U_0\lambda^2$. We see that l must be a function of η only.

Equations (2.9)—(2.11) are the seven equations determining the fourteen unknown functions F_1, F_2, f_0, f_1, f_2, f_3, f_4, h_0, h_1, h_2, h_3, h_4, h_5 and l. Hence the problem cannot be solved uniquely without introducing further conditions. Prof. Chou has increased the number of equations by constructing the equations of triple correlation. But the number of the unknown function is duely increased in turn owing to the introduction of the fourth order correlation functions on which other assumptions have to be made. In the present case since viscosity is not neglected, Chou's method cannot be followed for the resulting equations would be too complicated to be solvable.

It has been observed experimentally that the functions f_1, f_2, f_3 and f_4 vary very slowly with η near the central portion of the wake. We may expect the triple correlations to behave in like manner. Following C. C. Lin's treatment of the terbulent jet[4], we replace these correlations by their values near $\eta = 0$; apparently the deviation from fact will not be large. Thus h'_1, h'_3, and h'_5 are all constant and h'_0, h'_2, and h'_4 are all equal to zero, as h_1, h_3 and h_5 are odd functions of η while h_0, h_2 and h_4 are even functions of η. We may

(4) C. C. Lin, "Velocity and Temperature Distributions in Turbulent Jets," Sci. Rep. of National Tsing Hua Univ. (1941).

further assume that l is also a constant; this is equivalent to assuming that the lenth λ is constant across the wake. Then the number of unknown functions reduces to seven, satisfying the seven differential equations (2.9), (2.10) and (2.11).

From the third and fourth equations of (2.11) we get immediately, on integration.

$$f_3 = a - b\,\eta^k, \tag{2.12}$$

$$f_4 = a' - b'\eta^k, \tag{2.13}$$

where $a = -h'_3/(k_{(22)}\,l-1)$, $k=2(k_{(22)}\,l-1)$, $a'= -h'_5/(k_{(33)}\,l-1)$, $k'=2(k_{(33)}\,l-1)$ and b, b' are the two constants of integration. From the relation $\eta=y/\sqrt{x}$ we see that the value of η at a given point depends on the choice of the unit of length in terms of which x and y are measured. By the suitable choice of this unit we may adjust the scale of η such that $a=b$. Then (2.12) becomes

$$f_3 = a(1-\eta^k). \tag{2,14}$$

As an interesting experimental fact we note that if the unit of length is taken as the diameter of the cross-section of cylinder then $\eta \sim 1$ at the boundary nearly in all the existing observations (cf. referances 5,6,7,). From (2.8) and (2.14) we may determine the constant a directly from the relation

$$a = \frac{x}{DU_0^2}\,(\overline{w_y^2})_{y=0}$$

where D is the diameter of the cylinder.

A comparison of (2.14) and the experimental result (Reynolds number $=580$) of Fage[5] is shown in Fig. 1. It is seen that for his case the theoretical curve that fits experiment best is $k=4$ and

$a = x\overline{w_y^2}\,\Big|_{y=0} \Big/ DU_0^2 = 35 \times 0.09^2 \sim 0.3$; here the root-mean-

5. A. Fage, A. R. C. Report Memoranda, No. 1610 (1933).

square values of velocity fluctuations have been taken by Fage to be one third of their maximum values.

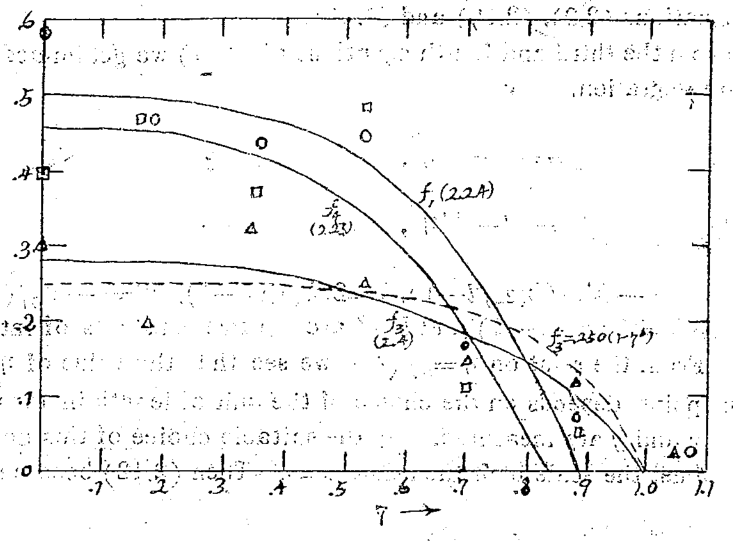


Fig. 1.—Velocity fluctuation distribution.
O: f_1; □: f_2; △: f_3 —Experiments by Fage.

Substituting (2.14) into the second equation of (2.11) and integrating, we obtain

$$\text{(2.15)} \qquad \log F_1 = \int \frac{-m\eta\, d\eta}{a(1-\eta^k)-\frac{1}{4}\eta^2} + \log C,$$

where $m=\frac{1}{3}k(12)\,l-\frac{3}{4}$ and C is the integration constant. Since experiment shows that the dynamical similarity also exists in the different wakes at different Reynolds numbers we must have $k=4$ universally. Then we obtain, on integration,

$$\text{(2.16)} \qquad F_1 = C\left[\frac{d-\eta^3-\frac{1}{8a}}{d+\eta^3+\frac{1}{8a}}\right]^{m/4ad}, \qquad \eta^2 \leqq d-\frac{1}{8a}$$

where $\bar{a}=\sqrt{1-1/8^3a^2}$; the inequality is introduced here since F_1 cannot be negative. This shows that the velocity distribution is narrower than the breadth of the fluctuation distribution.

Now the similarity of velocity distribution in different wakes requires either that a must be a universal constant for all Reynolds numbers (after the unit of length is chosen such that $\eta=1$ at the boundary of the wake), or that the effect of different values of a on the shape of the velocity distribution curve is small. The first condition is actually satisfied by Walker's experiment[6] which gives $\left(\sqrt{\overline{w_y^2}}\right)_{y=0}/U_0=0.12$, $x/D=29$ at the Reynolds number 6.9×10^6 and hence $a=29\times0.12^2\sim0.4$ which has the same order of magnitude as Fage's value $a\sim0.3$ at the Reynolds number 580. As for the second condition, the author has found from actual calculation that as long as a varies from 0.1 to 1.0, the form of the velocity distribution curve does not change by an amount greater than the experimental discrepancy allowed for the different wakes observed.

For the comparison with experiment we shall choose Fage and Falkner's data[7], since in their experiment the temperature distribution has also been carefully measured. In (2.12) the scale of η has been chosen such that $f_3=0$ at $\eta=1$, while in Fage and Falkner's experiment another variable ξ is used which gives $F_1=0$ at $\xi=1$. Since in our case $F_1=0$ at $\eta=\sqrt{\bar{a}-1/8a}$, the relation between ξ and η must be $\xi=\eta/\sqrt{\bar{a}-1/8a}$. But, as we shall see later from (3.10), the temperature distribution predicted by the present theory is a certain power of the velocity distribution function and hence also vanishes at $\eta=\sqrt{\bar{a}-1/8a}$, which corresponds experimentally to $\xi=1.25$ of Fage and Falkner's scale, so we introduce a new variable ζ and put $\zeta=1.25\eta/\sqrt{\bar{a}-1/8a}$. Then both the mean velocity and temparature distributions vanish at $\zeta=1$. Thus if we use $a=0.30$ and take from the experimental curve the value $F(0.41)/F(0)=0.205$ as a boundary condition to determine the constant $m/4ad$ in (2.16) to be 4.06, equation (2.16) becomes finally.

6. A. Fage, *Proc. Roy. Soc. Lond.* (A) *155* (1936), 592.
7. A. Fage and V. M. Falkner, *Proc. Roy. Soc. Lond.* (A) *135* (1932), 702.

$$(2.17)\qquad \frac{F_1(\eta)}{F_1(0)}=\left[2.25\,\frac{0.67-\eta^2}{1.51+\eta^2}\right]^{4.06}=\left[2.25\,\frac{1-\zeta^2}{2.25+\zeta^2}\right]^{4.06}$$

$$\eta^2=0.67\zeta^2.$$

The curve represented by (2.17) is plotted in Fig. 2 and tabulated in Table I for different values of ζ. It is seen that the agreement between theory and experiment is very good.

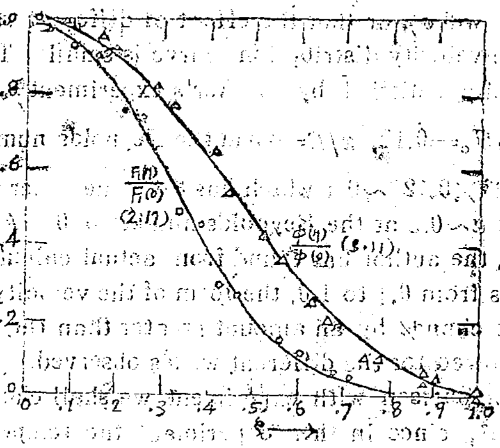


Fig. 2.—Velocity and temperature distributions behind an infinite cylinder. O: velocity; $\triangle$: temperature —Experiments by Fage and Falkner.

Table I

$\zeta=\eta\sqrt{0.67}$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$F_1(\eta)/F_1(0)$	1.00	.942	.791	.582	.374	.205	.089	.029	.009	.007	0
$\phi(\eta)/\phi(0)$	1.00	.973	.897	.777	.630	.475	.322	.191	.111	.025	0

We shall now proceed to determine the function f_1. From Fage's experiment we have approximately

$$(2.18)\qquad F_1(0)=0.2\times\sqrt{35.1}\cong1.2,$$

and from the first equation of (2.9) we have

$$(2.19)\qquad f_1=-0.6\eta\left[2.25\,\frac{0.67-\eta^2}{1.51+\eta^2}\right]^{4.06}$$

Differentiating (2.17) and using (2.18), we obtain

$$(2.20) \qquad F_1'(\eta) \cong -5.5\times10^2\, \eta\, \frac{(0.67-\eta^2)^3}{(1.51+\eta^2)^5}$$

Multiplying (2.20) by (2.19), we get

$$(2.21) \qquad f_2\, F_1' \cong 8.0\times10^3\, \eta\, \frac{(0.67-\eta^2)^7}{(1.51+\eta^2)^9}.$$

Calculation show that when $\eta \cong 0, 3$, the value of $f_2\, F'_1$ is a maximum and has the value 0.29; it decreases very rapidly, when η is slightly greater or less than 0.3. Thus when $\eta=0.2$, $f_2\, F_1'=0.025$; when $\eta=0.5$, $f_2 F_1'=0.0029$ and remains negligibly small when η is beyond these values. Then we can neglect the term $f_2\, F_1'$ in the first equation of (2.11) without causing a serious error, since its value is negligibly small over a large portion of the cross-section and still smaller than f_1 in the neighborhood of $\eta=0.3$. Therefore we have

$$f_1+\tfrac{1}{2}\eta f_1'-h_1'=k_{(11)}\,l\,f_1$$

This equation can be integrated in the same way as the third and fourth equations of (2.10), giving

$$(2.22) \qquad f_1=a''-b''\,\eta^{k''}$$

where $a''=-h_1'\,/\,(k_{(11)}\,l-1)$, $k''=2\,(k_{(11)}\,l-1)$ and b'' is the constant of integation.

Comparing (2.13) and (2.22) with Fage's experiment in Fig. 1 we have

$$(2.23) \qquad f_4=0.45(1-0.92\,\eta^4),$$

$$(2.24) \qquad f_1=0.50(1-0.83\,\eta^4).$$

From the form of the solutions (2.14), (2.23) and (2.24) we see that turbulence in the wake behind an infinite cylinber is three dimensional in general, though the mean velocity distribution is two dimensional. This is an experimental fact that has been observed by Fage already.

In the solutions (2.12), (2.13) and (2.22) for the mean squares of velocity fluctuations, we may also attempt to choose $k=k'=k''$ to be 6 instead of 4 to explain Fage's experimental data (Fig. 1). It would be interesting to investigate the effect of this change thus produced upon the velocity and temperature distributions. The author has used $k=6$ and integrated (2.15) for $a=0.3$ and $a=0.5$ and found respectively

$$(2.25)\qquad \frac{F_1(\eta)}{F_1(0)}=\left\{ 1.224\frac{1-\zeta^3}{[1.568+0.734(\zeta^2+\zeta^4)]^{1/2}} \exp[0.636-1.620\tan^{-1}0.364(2\zeta^2+1)]\right\}^{2.43}$$

$$(a=0.3,\quad \zeta=\eta/\sqrt{0.837})$$

$$(2.26)\qquad \frac{F_1(\eta)}{F_1(0)}=\left\{ 1.092\frac{1-\zeta^3}{[1.191+0.691(\zeta^2+\zeta^4)]^{1/2}} \exp[0.483-1.236\tan^{-1}0.419(2\zeta^2+1)]\right\}^{2.78}$$

$$(a=0.5,\quad \zeta=\eta/\sqrt{0.831})$$

The above two formulae are tabulated in Table II. Their agreement with the former case, $k=4$ and $a=0.3$ is excellent. This shows that a slight variation of both the shape and magnitude of the fluctuation distribution curve has practically no effect on the mean velocity and also on the temperature distributions as we shall see in the following section.

Table II

ζ		0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\frac{F_1(\eta)}{F_1(0)}$	$a=0.3$	1.00	.917	.828	.630	.471	.205	.180	.039	.032	.003	0
	$a=0.5$	1.00	.940	.752	.601	.363	.205	.094	.034	.009	.001	0

3. *The two-dimensional wake behind an infinite cylinder*
Temperature distribution.

In order to examine Chou's theory more closely, we shall now consider the distribution of temperature across the wake behind a heated infinite cylinder. Since the fundamental equation for the conduction of heat inside a fluid is exactly similar to the Navier-Stokes' equations of motion except that (1) there is no term that corresponds to the pressure term in the temperature case and (2) the kinematic coefficient of viscosity ν is replaced by the thermal diffusivity κ, we may consider the temperature as the fourth component of velocity. Thus replacing U_k and $w_{(i)}$ respectively by $\bar{\theta}$, the mean temperature, and θ', the temperature fluctuation, then equation (1.1) and (1.2) become respectively

$$(3.1) \qquad U^j \bar{\theta}_{,j} = -\overline{w^j \theta'_{,j}} + \kappa \nabla^2 \bar{\theta},$$

$$(3.2) \qquad U_{i,j}\overline{\theta' w^j} + \bar{\theta}_{,j}\overline{w_i w^j} + U^j(\overline{w_i \theta'})_{,j} + (\overline{w^j w_i \theta'})_{,j}$$

$$= -\frac{1}{\rho}\overline{w_{,i}\theta'} + \nu\overline{\theta' \nabla^2 w_i} + \kappa\overline{w_i \nabla^2 \theta'}.$$

Applying equation (3.1) to the two-dimensional wake and neglecting the term, $\kappa \nabla^2 \bar{\theta}$ we have

$$(3.3) \qquad U\frac{\partial \bar{\theta}}{\partial x} + V\frac{\partial \bar{\theta}}{\partial y} = -\frac{\partial}{\partial x}\overline{w_x \theta'} - \frac{\partial}{\partial y}\overline{w_y \theta'}$$

which corresponds to (2.1). If we put in analogy to the last section

$$(3.4) \qquad \nu\overline{\theta' \nabla^2 w_i} + \kappa\overline{w_i \nabla^2 \theta'} = -\frac{2\nu_1 k_{(i)}}{\lambda^2}\overline{\theta' w_i}$$

where ν_1 is a certain linear function of ν and κ, $k_{(i)}$ are constants and has the same meaning as before. Neglecting the terms $\overline{w_i \theta'}$ in (3.2), we have

$$
(3.5)\quad \left\{
\begin{aligned}
&\overline{w_x\theta'}\frac{\partial V}{\partial x}+\overline{w_y\theta'}\frac{\partial V}{\partial y}-\frac{1}{\rho}\left(\tau_{xx}\frac{\partial\bar\Theta}{\partial x}+\tau_{xy}\frac{\partial\bar\Theta}{\partial y}\right)\\
&\quad +U\frac{\partial}{\partial x}\overline{w_x\theta'}+V\frac{\partial}{\partial y}\overline{w_x\theta'}+\frac{\partial}{\partial x}\overline{w_x^{2}\theta'}\\
&\quad +\frac{\partial}{\partial y}\overline{w_xw_y\theta'}=-\frac{2\nu_1k_{(1)}}{\rho\lambda^2}\overline{\theta'w_x}\\
&\overline{w_x\theta'}\frac{\partial V}{\partial x}+\overline{w_y\theta'}\frac{\partial V}{\partial y}-\frac{1}{\rho}\left(\tau_{yx}\frac{\partial\bar\Theta}{\partial x}+\tau_{yy}\frac{\partial\bar\Theta}{\partial y}\right)\\
&\quad +U\frac{\partial}{\partial x}\overline{w_y\theta'}+V\frac{\partial}{\partial y}\overline{w_y\theta'}+\frac{\partial}{\partial x}\overline{w_xw_y\theta'}\\
&\quad +\frac{\partial}{\partial y}\overline{w_y^2\theta'}=-\frac{2\nu_2k_{(2)}}{\rho\lambda^2}\overline{\theta'w_y}
\end{aligned}
\right.
$$

the third equation ($i=3$) of (3.2) being identically zero.

Following the same reasoning as before and since the integral

$$U_0\int_{-\infty}^{\infty}(\bar\Theta-\bar\Theta_0)\,dy$$

which is proportional to the heat lost by the cylinder per unit length per unit time, must be independent of x, then similar to (2.3) we may put

$$(3.6)\qquad \bar\Theta_0-\bar\Theta=\frac{U_0}{\sqrt{x}}\,\phi(\eta),\qquad \eta=\frac{y}{\sqrt{x}}$$

where $\bar\Theta_0$ is the uniform temperature outside the wake. Let

$$\overline{w_x\theta'}=\frac{U_0^{2}}{x}f_5(\eta)\qquad \overline{w_y\theta'}=\frac{U_0^{2}}{x}f_6(\eta)$$

$$\overline{w_x^{2}\theta'}=\frac{U_0^{3}}{x^{3/2}}h_6'(\eta),\quad \overline{w_xw_y\theta'}=\frac{U_0^{3}}{x^{3/2}}h_7(\eta),$$

$$\overline{w_y^{2}\theta'}=\frac{U_0^{3}}{x^{3/2}}h_8(\eta)$$

If we retain the small terms of the lowest order, then (3.3), after being integrated with respect to η once, and (3.5) respectively reduce to

$$(3.7)\qquad \eta\phi = -2f_6$$

$$(3.8)\qquad \begin{aligned} F'_1 f_6 + \phi' f_2 + f_5 + \tfrac{1}{2}\eta f'_5 - h'_7 &= k_{(1)} l_1 f_5 \\ \phi' f_3 + f_6 + \tfrac{1}{2}\eta f'_6 - h'_8 &= k_{(2)} l_1 f_6 \end{aligned}$$

where $l_1 = 2\nu_1 x/U_0\lambda^2$

Since h_8 is an even function of η, h'_8 must be an old function. Hence following C. C. Lin[4], we may put $h'_8 = 0$. Substituting (3.7) into the second equation of (3.8), then we have

$$(f_3 - \tfrac{1}{4}\eta^2)\phi' + (\tfrac{1}{2}k_{(2)}l_1 - \tfrac{2}{4})\eta\phi = 0$$

Using (2,14), we obtain

$$(3.9)\qquad \log\phi = \int \frac{m_1\eta\, d\eta}{a(1-\eta^k) - \frac{1}{4}\eta^2} + \log B = \frac{m_1}{m}\log F_1 + \log\frac{B}{C}$$

where $m_1 = \frac{1}{2}k_{(2)}l - \frac{2}{4}$ and $\log B$ is a constant of integration. Then by eliminating F_1 with (2.16), we find

$$(3.10)\qquad \phi = \frac{B}{C}F_1^{\frac{m_1}{m}} = B\left[\frac{d-\eta^2-\frac{1}{8a}}{d+\eta^2+\frac{1}{8a}}\right]^{\frac{m_1}{4ad}}$$

It it seen that $\phi = 0$ at the same point where $F_1 = 0$. The experiment of Fage and Falkner gives $\phi(0.41) = 0.45$. Substituting the constant $a = 0.30$ into (3.10) as before, we have

$$(3.11)\qquad \frac{\phi(\eta)}{\phi(0)} = \left[2.25\,\frac{0.67 - \eta^2}{1.51 + \eta^2}\right]^{1.91} = \left[2.25\,\frac{1 - \xi^2}{2.25 + \xi^2}\right]^{1.91}$$

in which the value of $m_1/4ad = 1.91$ is determined from the boundary conditon of ϕ at $\eta = 0.41$. Comparison of the theoretical curve plotted by means of (3.11) which is also tabulated in Table I for different values of η which Fage and Falkner's experiment date is also shown in Fig. 2. We see that the agreement is very satisfactory.

The temperature diatribution behind a heated infinite cylinder has also been calculated for the cases $k = 6$, $a = 0.3$ and $a = 0.5$ separately. For both cases we find

$$(8.12)\qquad \frac{\phi(\eta)}{\phi(0)} = \left[\frac{F_1(\eta)}{F_1(0)}\right]^{0.47}$$

where $F_1(\eta)/F_1(0)$ is given by (2.25) and (2.26) for $a = 0.3$ and $a = 0.5$ respectively. For both cases agreement with observation is good.

4. *The axially symmetrical wake behind a body of revolution: Velocity and temperature distributions.*

The method of solution in § 2 applies immediately to the present case of three-dimensional wake except that the calculation becomes now slightly more complicated due to the form of equations of mean motion and double correlations in cylindrical coordinates. Let us take the x-axis as the axis of symmetry of the body of revolution which lies in the direction of the undisturbed velocity of flow outside the wake and the point $x=0$ somewhere inside the body. Denote by U and V the components of mean velocity in the x and r directions respectively and write $x=x^1$, $r=x^2$ and $\theta=x^3$, then (1.1) and (1.2) under the same modifications as mentioned in the first paragragh of § 2 can be written in the following form

$$
(4.1)\quad \left\{
\begin{aligned}
& U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial r}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_{xx}}{\partial x}+\frac{1}{\rho}\frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{xr}) \\
& U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial r}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial r}+\frac{1}{\rho}\frac{\partial \tau_{rx}}{\partial x}+\frac{1}{\rho}\frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rr}) \\
& \qquad -\frac{1}{\rho}\frac{1}{r}\tau_{\theta\theta}
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
& -\frac{2}{\rho}\left(\frac{\partial U}{\partial x}\tau_{xx}+\frac{\partial U}{\partial r}\tau_{xr}\right)-\frac{1}{\rho}U\frac{\partial \tau_{xx}}{\partial x}-\frac{1}{\rho}V\frac{\partial \tau_{xx}}{\partial r} \\
& \qquad +\frac{\partial}{\partial x}\overline{w_x{}^3}+\frac{\partial}{\partial r}\overline{w_x{}^2 w_r}+\frac{1}{r}\overline{w_x{}^2 w_r}=\frac{2\nu k_{(11)}}{\rho\lambda^2}\tau_{xx}
\end{aligned}
\right.
$$

$$
(4.2)\quad \left\{
\begin{aligned}
& -\frac{1}{\rho}\left[\frac{\partial V}{\partial x}\tau_{xx}+\frac{\partial U}{\partial r}\tau_{rr}+\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial r}\right)\tau_{rx}\right]-\frac{1}{\rho}U\frac{\partial \tau_{xr}}{\partial x} \\
& \qquad -\frac{1}{\rho}V\frac{\partial \tau_{xr}}{\partial r}+\frac{\partial}{\partial x}\overline{w_x{}^2 w_r}+\frac{\partial}{\partial r}\overline{w_x w_r{}^2}-\frac{1}{r}\overline{w_x w_\theta{}^2} \\
& \qquad +\frac{1}{r}\overline{w_x w_r{}^2}=\frac{2\nu k_{(12)}}{\rho\lambda^2}\tau_{xr}, \\
& -\frac{2}{\rho}\left(\frac{\partial V}{\partial x}\tau_{xr}+\frac{\partial V}{\partial r}\tau_{rr}\right)-\frac{1}{\rho}U\frac{\partial \tau_{rr}}{\partial x}-\frac{1}{\rho}V\frac{\partial \tau_{rr}}{\partial r} \\
& \qquad +\frac{\partial}{\partial x}\overline{w_x w_r{}^2}+\frac{\partial}{\partial x}\overline{w_r{}^3}-\frac{2}{r}\overline{w_r w_\theta{}^2}+\frac{1}{r}\overline{w_r{}^3}
\end{aligned}
\right.
$$

$$\left\{\begin{array}{l} \quad = \dfrac{2\nu k_{(22)}}{\rho\lambda^2}\tau_{rr} \\ -\dfrac{1}{\rho}U\dfrac{\partial\tau_{\theta\theta}}{\partial x} - \dfrac{1}{\rho}V\dfrac{\partial\tau_{\theta\theta}}{\partial y} + \dfrac{\partial}{\partial x}\overline{w_x w_\theta{}^2} \\ \quad + \dfrac{1}{r^3}\dfrac{\partial}{\partial r}\left(r^3\overline{w_\theta{}^2 w_r}\right) = \dfrac{2\nu k_{(33)}}{\rho\lambda^2}w_{\theta\theta} \end{array}\right.$$

where τ_{xr}, $\tau_{\theta\theta}$, etc., correspond to τ_{jk}, and $\overline{w_x w_r{}^2}$, $\overline{w_x w_\theta{}^2}$, etc., to $\overline{w_x w_j w^k}$ in tensor lanuage. The difference between w_3 and w^3 must be noted here, since in cylindrical coordinates $w_3 = r^2 w^3$. In writing(4.1)and(4.2) the terms which just vanish identically according to the symmetry property of the wake have already been dropped.

Now similar to the two-dimensional wake experiment has shown that (1) there is dynamical similarity in different cross-sections of the wake at large distance down flow, (2) the boundary of the wake is found to be a surface of revolution, r = const. $x^{1/3}$, and (3) the dragging force given by the integral

$$2\pi\rho U_0 \int_0^\infty (U_0 - U)\, r\, dr$$

in which U_0 denotes the value of U outside the wake, must be independent of the position of the plane of integration behind the obstacle. Then we put

$$(4.3)\qquad U = U_0\left[1 - \frac{1}{x^{2/3}}F_1(\eta)\right], \quad \eta = r/x^{1/3}$$

From the equation of continuity,

$$(4.4)\qquad \frac{\partial U}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rV) = 0$$

we have

$$(4.5)\qquad V = \frac{U_0}{x^{4/3}}F_2(\eta)$$

F_1 and F_2 are two unknown functions to be determined presently.

Exactly as before we put $\varepsilon = 1/x^{1/3}$ as a small quantity of the first order; then we get from differentiation

$$(4.6)\quad \begin{cases} U_0 - U \sim \varepsilon^2, & \dfrac{\partial U}{\partial x} \sim \varepsilon^5, \quad \dfrac{\partial U}{\partial r} \sim \varepsilon^3 \\ V \sim \varepsilon^4, & \dfrac{\partial V}{\partial x} \sim \varepsilon^7, \quad \dfrac{\partial V}{\partial r} \sim \varepsilon^5. \end{cases}$$

Let the fluctuation component w_i be proportional to ε^s. Neglecting all terms which are small quantities of order higher than the fifth in the first equation of (4.1), we find

$$\rho U_0 \frac{\partial U}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{xr}) \sim \varepsilon^5$$

Hence

$$(4.7)\quad s = 2.$$

Therefore we can put

$$(4.8)\quad \begin{cases} \dfrac{1}{\rho}\,\overline{p} = \dfrac{U_0^{\,2}}{x^{4/3}} f_0(\eta), & -\dfrac{1}{\rho}\tau_{xx} = \dfrac{U_0^{\,2}}{x^{4/3}} f_1(\eta), \\ -\dfrac{1}{\rho}\tau_{xr} = \dfrac{U_0^{\,2}}{x^{4/3}} f_2(\eta), & -\dfrac{1}{\rho}\tau_{rr} = \dfrac{U_0^{\,2}}{x^{4/3}} f_3(\eta), \\ -\dfrac{1}{\rho}\tau_{\theta\theta} = \dfrac{U_0^{\,2}}{x^{4/3}} f_4(\eta), & \\ \overline{w_x^{\,3}} = \dfrac{U_0^{\,3}}{x^2} h_0(\eta), & \overline{w_x^{\,2} w_r} = \dfrac{U_0^{\,3}}{x^2} h_1(\eta), \\ \overline{w_x w_r^{\,2}} = \dfrac{U_0^{\,3}}{x^2} h_2(\eta), & \overline{w_r^{\,3}} = \dfrac{U_0^{\,3}}{x^2} h_3(\eta), \\ \overline{w_x w_\theta^{\,2}} = \dfrac{U_0^{\,3}}{x^2} h_4(\eta), & \overline{w_r w_\theta^{\,2}} = \dfrac{U_0^{\,3}}{x^2} h_5(\eta) \end{cases}$$

Subtituting these relations into (4.1), the first equation of which is then integrated once with respect to η, (4.2), and (4.4) and neglecting all terms except those of the lowest order of magnitudes in each equation, we obtain

$$(4.9)\quad \eta F_1 = -3 f_2; \qquad f'_0 = f'_3 + \frac{1}{\eta} f_3 - \frac{1}{\eta} f_4;$$

$$(4.10)\quad \eta F_1 = -3 F_2;$$

$$
(4.11)\quad \begin{cases} F'_1 f_2 + \dfrac{4}{3} f_1 + \dfrac{1}{3}\eta f'_1 - h'_1 - \dfrac{1}{\eta} h_1 = k_{(11)}\, l f_1 \\ F'_1 f_3 + \dfrac{4}{3} f_2 + \dfrac{1}{3}\eta f'_2 - h'_2 - \dfrac{1}{\eta} h_4 + \dfrac{1}{\eta} h_2 = k_{(12)}\, l f \\ \dfrac{4}{3} f_3 + \dfrac{1}{3}\eta f'_3 - h'_3 + \dfrac{2h_5}{\eta} - \dfrac{h_3}{\eta} = k_{(22)}\, l f_3 \\ \dfrac{4}{3} f_4 + \dfrac{1}{3}\eta f'_4 - h'_5 - \dfrac{3}{\eta} h_5 = k_{(33)}\, l f_4 \end{cases}
$$

where $l = 2\nu\, x^{2/3}/\lambda^2 U_0$ as before. Comparing (4.11) with (2.11), we see that they are exactly similar except the additional terms arisen out of the curvature of the cylindrical coordinates.

Similar to the two-dimensional case we may substitute the functions h'_i by their values near $\eta = 0$. Thus h'_1, h'_3 and h'_5 are all constants and h'_0, h'_2 and h'_4 are all equal to zero. Furthermore from the second equation of (4.11) we have

$$
(4.12)\qquad h_2 - h_4 = 0
$$

when $\eta = 0$. Therefore according to our assumption (4.12) must also hold throughout the whole cross-section of the wake. Then the third equation of (4.11) gives, on integration:

$$
(4.13)\qquad f_3 = a - b\eta^{k}
$$

where $a = -(2h'_3 - h'_5)/(k_{(22)}\, l - \frac{4}{3})$, for we have replaced h_3 and h_5 approximately by $h'_3\eta$ and $h'_5\eta$ respectively, $k = 3(k_{(22)}\, l - \frac{4}{3})$ and b is a constant of integration. We may as before choose the unit of length such that $a = b$ and consequently (4.3) becomes

$$
(4.14)\qquad f_3 = a(1 - \eta^{k}).
$$

Substituting this into the second equation of (4,11) and integrating, we have

$$
(4.15)\qquad \log F_1 = \int \frac{-m\eta\, d\eta}{a(1-\eta^{k}) - \frac{1}{9}\eta^2} + \log C
$$

where $m = \frac{1}{3} k(12) l - \frac{5}{9}$ and C is another constant of integration.

No experimental measurement of mean squares of velocity fluctuations in the axially symmetrical wake seems to have been made. Therefore the value of k cannot be determined from the comparison with experiment. However, it is reasonable to use $k \doteqdot 4$ obtained from the two-dimensional wake. Then (4.15) can be integrated as before and we have

$$(4.16)\qquad F_1(\eta) = C\left[\frac{d - \eta^2 - \frac{1}{18a}}{d + \eta^2 + \frac{1}{18a}}\right]^{\frac{m}{4ad}}, \quad \eta^2 \leqq d - \frac{1}{18a}$$

where $d = \sqrt{1 + 1/18^2 a^2}$.

Next we shall consider the temperature distribution in axially symmetrical wake. The equations corresponding to (3.3) and (3.5) of the last section become

$$(4.17)\qquad U\frac{\partial\overline{\theta}}{\partial x} + V\frac{\partial\overline{\theta}}{\partial r} = -\frac{\partial}{\partial x}\overline{w_x\theta'} - \frac{1}{r}\frac{\partial}{\partial r}(r\overline{w_r\theta'}) ;$$

$$(4.18)\qquad \left\{\begin{aligned} &\overline{w_x\theta'}\frac{\partial U}{\partial x} + \overline{w_r\theta'}\frac{\partial U}{\partial r} - \frac{1}{\rho}\left(\tau_{xx}\frac{\partial\overline{\theta}}{\partial x} + \tau_{xr}\frac{\partial\overline{\theta}}{\partial r}\right) + U\frac{\partial}{\partial x}\overline{w_x\theta'} \\ &\qquad + V\frac{\partial}{\partial r}\overline{w_x\theta'} + \frac{\partial}{\partial x}\overline{w_x^2\theta'} + \frac{1}{r}\frac{\partial}{\partial r}(r\,\overline{w_x w_r w_\theta'}) \\ &\qquad = -\frac{2\nu_1 k(1)}{\rho\lambda^2}\overline{w_x\theta'} \\ &\overline{w_x\theta'}\frac{\partial V}{\partial x} + \overline{w_r\theta'}\frac{\partial V}{\partial r} - \frac{1}{\rho}\left(\tau_{xr}\frac{\partial\overline{\theta}}{\partial x} + \tau_{rr}\frac{\partial\overline{\theta}}{\partial r}\right) + U\frac{\partial}{\partial x}\overline{w_r\theta'} \\ &\qquad + V\frac{\partial}{\partial r}\overline{w_r\theta'} + \frac{\partial}{\partial x}\overline{w_x w_r\theta'} + \frac{1}{r}\frac{\partial}{\partial r}(r\overline{w_r^2\theta'}) \\ &\qquad - \frac{1}{r}\overline{w_\theta^2\theta'} = -\frac{2\nu_1 k(2)}{\rho\lambda^2}\overline{w_r\theta'} \end{aligned}\right.$$

the equation ($i=3$) being identically zero.

Since the integral

$$2\pi\int_0^\infty U_0(\overline{\theta} - \overline{\theta}_0)\, r\,dr$$

which is proportional to the heat lost by the obstacle per unit time must be independent of x, then as an analogy to (4.3) and (4.8), we may put

$$
(4.19)\quad \begin{cases}
\bar{\theta}_0-\bar{\theta}=\dfrac{U_0}{x^{2/3}}\phi(\eta), & \eta=r/x^{1/3} \qquad l_1=2\nu_1 x^{2/3}/\lambda^2 U_0 \\
\overline{w_x\theta'}=\dfrac{U_0^{\,2}}{x^{4/3}}f_5(\eta), & \overline{w_r\theta'}=\dfrac{U_0^{\,2}}{x^{5/3}}f_6(\eta) \\
\overline{w_x^{\,2}\theta'}=\dfrac{U_0^{\,3}}{x^2}h_6(\eta), & \overline{w_x w_r\theta'}=\dfrac{U_0^{\,3}}{x^3}h_7(\eta) \\
\overline{w_r^{\,2}\theta'}=\dfrac{U_0^{\,3}}{x^2}h_8(\eta), & \overline{w_\theta^{\,2}\theta'}=\dfrac{U_0^{\,3}}{x^2}h_9(\eta)
\end{cases}
$$

Thus, equations (4.17) and (4.18) become, on neglecting small quantities of higher order in each equation,

$$(4.20)\qquad \eta\phi=-3f_6,$$

and

$$
(4.21)\quad \begin{cases}
F'_1 f_6+\phi' f_2+\dfrac{4}{3}f_5+\dfrac{1}{3}\eta f'_5-h'_7-\dfrac{1}{\eta}h_7=k_{(1)}\,l_1\,f_5 \\
\phi' f_3+\dfrac{4}{3}f_6+\dfrac{1}{3}\eta f'_6-h'_8-\dfrac{1}{\eta}h_8+\dfrac{h_9}{\eta}=k_{(2)}\,l_1\,f_6
\end{cases}
$$

respectively.

Following the same way as before, we may assume h'_7, h_8 and h_9 as constants. Since $\phi'(0)=0$, and $f_6(0)=0$, from the second equation of (4.21) we find, by setting $\eta=0$,

$$(4.22)\qquad h'_8=0,\quad h_8-h_9=0.$$

Substituting (4.22) back into the second equation of (4.21), we have

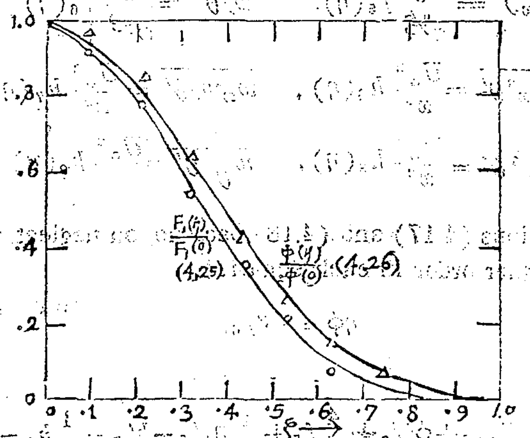
$$(4.23)\qquad \phi' f_3+\frac{4}{3}f_6+\frac{1}{3}\eta f'_6=k_{(2)}\,l\,f_6$$

We see that (4.20) and (4.33) have exactly the same forms as the first equation of (4.9) and the second equation of (4.11). Therefore on integration we have

$$(4.24)\qquad \phi(\eta)=B\left[\frac{d-\eta^2-\frac{1}{18a}}{d+\eta^2+\frac{1}{18a}}\right]^{\frac{m_1}{4ad}}$$

where $m_1=\frac{1}{3}k_{(2)}\,l-\frac{5}{9}$ and $d=\sqrt{1+1/18^2a^2}$.

Fig. 3 shows the experimental data of velocity and temperature



distributions obtained by Hall and Hislop[8]. We may determine d by giving a reasonable value of a and then get the value of m and m_1 from the experimental value of F_1 and φ at $\eta=0.41$. Using the value of $a=0.6$, we find that (4.16) and (4.24) become respectively

$$(4.25)\qquad \frac{F_1(\eta)}{F_1(0)}=\left[1.22\,\frac{0.90-\eta^2}{1.10+\eta^2}\right]^{3.50}=\left[1.22\,\frac{1-\zeta^2}{1.22+\zeta^2}\right]^{3.50}$$

$$(\zeta=\eta/\sqrt{0.90})$$

$$(4.26)\qquad \frac{\phi(\eta)}{\phi(0)}=\left[1.22\,\frac{0.90-\eta^2}{1.10+\eta^2}\right]^{2.86}$$

which are also tabulated in Table III for different values of ζ.

Table III

ζ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$F_1(\eta)/F_1(0)$	1.00	.938	.777	.550	.351	.190	.085	.030	.003	.001	0
$\phi(\eta)/\phi(0)$	1.00	.951	.817	.625	.480	.260	.135	.077	.017	.004	0

8. A. A. Hall and G. S. Hislop, *Proc. Camb. Phil. Soc.* 34 (1938), 48.

Here we must note that the boundary of the distribution of $\overline{w_r^2}$, $\eta=1$ (cf. (4.14)), has been chosen at $r/R=3$ where r is the distance from the axis of symmetry of the wake, and R is the value of r at which the velocity deficiency is just half the value at the center of the wake. Since no observation on the distribution of $\overline{w_r^2}$ has yet been made, the actual point where $\eta=1$ is not yet known. Hence the above chosen value of r/R for the boundary of the $\overline{w_r^2}$ distribution is quite arbitrary. It is seen from Fig. 3 that the agreement between theory and experiment is quite satisfactory in general, though there is marked deviation near the edge. This may be due to the fact that the chosen value of r/R for the boundary is not accurate. The author has tried the different values of a between 0.4 and 0.8 and then calculated the velocity and temperature distributions. It is found that the resulting changes in the shape of the curves lie within experimental error.

In both cases of the two-dimensional and the three-dimensional wakes we have found from the present theory that the breadth of the velocity and temperature distributions is narrower than that of the mean squares of velocity fluctuation ($\overline{w_y^2}$ or $\overline{w_r^2}$) distribution by an amount which is a function of a only. This is an important point which may be examined by later experiments. It will also be noted that in order to calculate the velocity and temperature distributions more accurately, observations of the velocity, temperature and fluctuation distributions should be performed in a single experiment at the same Reynolds number.

5. *The constant $k(ij)$ and their relation with the statistical theory of isotropic turbulence.*

In the foregoing treatment we have introduced a set of constants $k(ij)$ which have the value 5 for $i=j$ and are indeterminate for $i\neq j$ when the turbulence is isotropic. Since Chou's equations must hold equally good when the turbulence is isotropic, the value of $k(ij)$ in our problem must be the same as those determined in isotropic turbulence.

Then we have

9. G. I. Paylor, *Proc. Roy. Soc. Lond.* (A), 151 (1935), 421-478.

(5.1) $$k_{(11)} = k_{(22)} = k_{(33)} = 5.$$

To find the value of $k_{(12)}$ we solve the following equations for the two-dimensional wake:

(5.2) $$\begin{cases} k = 2(k_{(22)}\, l - 1), \quad k' = 2(k_{(33)}\, l - 1), \quad k'' = 2(k_{(11)}\, l - 1), \\ m = \dfrac{1}{2}\, k_{(12)}\, l - \dfrac{3}{4} \end{cases}$$

in which k, k', k'' and m have already been determined experimentally in § 2. For the three-dimensional wake the corresponding equations are

(5.3) $$\begin{cases} k = 3\left(k_{(22)}\, l - \dfrac{4}{3}\right), \quad k' = 3\left(k_{(33)}\, l - \dfrac{4}{3}\right), \\ k'' = 3\left(k_{(11)}\, l - \dfrac{4}{3}\right), \quad m = \dfrac{1}{3}\, k_{(12)}\, l - \dfrac{5}{9}, \end{cases}$$

in which k, k', k'' and m have also been determined in § 4. That (5.2) and (5.3) are consistent with (5.1) follows immediately from the experimental result;

(5.4) $$k = k' = k'' = 4.$$

From the solution of mean velocity distributions within the two-dimensional and three-dimensional wakes (cf. (2.16) and (4.16)) we see that if the constant a is varied, the value of η where F vanishes is also changed. Then in order to make (2.16) to fit into the experimental data we have to vary the power, $m/4ad$. Consequently the constant m depends upon a and on account of (5.3) $k_{(12)}$ can be regarded also as a function of a. The author has plotted different curves of F_1 for the different values of a with the corresponding values of $m/4ad$ and found all such theoretical curves agree with the experimental data very well. In the third column of Table IV the quantity, $m/4ad$, thus determined for the different values of a is given. It is seen from the last column that the empirical relation connecting m and a can be written as:

$$\frac{m}{4ad}=\frac{0.304}{ad}+3.12. \tag{5.5}$$

Then using the last equation of (5.2), we have

$$k_{(12)}\,l=2.3+25ad. \tag{5.6}$$

Table IV

a	0.1	0.3	0.5	0.7	0.9	∞
d	1.60	1.09	1.03	1.02	1.01	1.00
$m/4ad$	5.02	4.06	3.71	3.55	3.46	3.12
$\frac{m}{4ad}-3.12$	1.90	0.94	0.59	0.43	0.34	0.00
$ad\left(\frac{m}{4ad}-3.12\right)$	.304	.307	.304	.307	.309	—

Since experiment shows that $k=4$, then from (5,1) and the first equation of (5.2), we have $l\cong 0.60$. Substituting this value of l into (5.6), we find

$$k_{(12)}=3.9+42ad=3.9+42\sqrt{a^2+1/8^2}\,. \tag{5.7}$$

It is seen from the above equation that $k_{(12)}$ decreases as a decreases. The smallest possible value of $k_{(12)}$ will be 9.2. For $a=0.3$ and 0.5 the values of $k_{(12)}$ are 20 and 25 respectively. Thus we see that according to the present theory $k_{(12)}>k_{(11)}$, $k_{(22)}$ and $k_{(33)}$.

The above result shows that in wakes the turbulence has the tendency to become isotropic. Let us consider the case in which the mean fluid velocity is nearly constant everywhere and also along the direction of x which is taken as its direction of mean motion so that turbulence is also nearly uniform in the y and z directions but decays in the direction of mean motion (such as that exists in a wind tunnel). Then (1.2) can be written:

$$(5.6)\quad\left\{\begin{aligned}&-\frac{d\tau_{xx}}{dx}=\frac{2\nu k_{(11)}}{U\lambda^2}\tau_{xx}, &&-\frac{d\tau_{xy}}{dx}=\frac{2\nu k_{(12)}}{U\lambda^2}\tau_{xy},\\ &-\frac{d\tau_{yy}}{dx}=\frac{2\nu k_{(22)}}{U\lambda^2}\tau_{yy}, &&-\frac{d\tau_{zz}}{dx}=\frac{2\nu k_{(33)}}{U\lambda^2}\tau_{zz}.\end{aligned}\right.$$

It is seen from the above equations that if $k_{(12)}$ is larger than $k_{(ii)}$, then τ_{xy}, which was originally different from zero, will be decaying more rapidly than τ'_{xx}, τ'_{yy} and τ'_{zz}. This means that turbulence tends to become isotropic as the distance from the turbulence producing device increases.

The inequality $k_{(12)} > k_{(ii)}$ leads to another approximate form of solution of the equations of double correlations (2.2) and (4.2). Let us consider the second equation of (2.11). Since $k_{(12)}$ is much greater than unity, the second and third terms on the left-hand side of that equation are negligible when compared with the term $k_{(12)}lf_2$ on the right-hand side. Calculation also shows that the terms $f_3F'_1$ is much greater than f_2. Hence this equation can be written as:

$$f_3F'_1 = k_{(12)}\,lf_2. \tag{5.9}$$

Using (2.8) and (2.3), we have

$$\tau'_{xy} = -\frac{\lambda^2\tau'_{yy}}{2\nu k_{(12)}}\frac{\partial U}{\partial y}. \tag{5.10}$$

Since τ'_{yy} is essentially constant within the breadth of the velocity deficiency distribution, we may write (5.10) in the following form:

$$\tau'_{xy} = \mu_a\frac{\partial U}{\partial y}, \qquad \mu_a = \mu_a(x) = \frac{\lambda^2}{2\nu k_{(12)}}\,\rho\,\overline{w_y^2}. \tag{5.11}$$

We thus see that the present consideration leads to a formula of apparent stress τ'_{xy} which, like the viscous stress, is a linear function of $\partial U/\partial y$. The author has used this formula and the Reynolds equation of mean motion to solve the problem treated in § 2 and § 4 and found that for both the two-dimensional wakes F_1 is of the following form,

$$F_1(\eta) = A\,e^{-k\eta^2} \tag{5.12}$$

where A and k are constants. The agreement between (5.12) and the experimental data is also found to be satisfactory. The formula (5.11) for the apparent stress also holds good approximately according to C. C. Lin[4] in his solution of the velocity and temperature distributions in turbulent jets which also agree very well with the experiment.

We note from the foregoing discussion that there exists actually an apparent stress coefficient of viscosity μ_a in the field of turbulence which can be shown to be several hundred times great the ordinary coefficient of viscosity. This point is very interesting in meteorology, since the large viscosity (usually several hundred times that of air in laminar motion) of the atmospheric air may thus be explained by the existence of turbulence.

From the calculated value of l we can also calculate λ, using the relation which has been put forth in § 2 and § 4:

$$l = 2\nu x^q/\lambda^2 U_0, \tag{5.13}$$

where $q=1$ or $2/3$ according to the wake is two or three-dimensional. From the equation (5.4) and the first equations of (5.2) and (5.3) we obtain

$$l = 0.60 \text{ for the two-dimensional wake;} \tag{5.14}$$

$$l = 0.53 \text{ for the three-dimensional wake.} \tag{5.15}$$

In the experiment of Fage and Falkner, $U_0 = 55.6$ ft/sec, $x = 1.16$ ft., and $\nu = 0.000159$ ft/sec. Substituting these values and l from (5.17) into (5.16) with $q=1$, we find $\lambda = 0.001$ ft., which is about 0.005 of the breadth of the wake.

Now from (2.8) and (4.8) we can write

$$\frac{}{\overline{w_y^2}}\Big|_{y=0} = \overline{w_{y_0}^2} = \frac{U_0^2}{x^q} f_3(0) = \frac{U_0^2}{x^q} a. \tag{5.16}$$

Take the square root of both sides of (5.16) and eliminate U_0^2 by means of (5.15). Then

$$l = 2\nu a x^{q/2} / (\lambda^2 \sqrt{\overline{w_{y_0}^2}}). \tag{5.17}$$

Let y_0 be the distance from the center of the wake where $\eta = \frac{w}{2}$. So

$$y_0 = \tfrac{1}{2} x^{q/2}, \tag{5.18}$$

and (5.17) can be written in the following form,

$$\frac{\lambda^2}{y_\circ^2}=\frac{4a}{l}\frac{\nu}{y_\circ\sqrt{\overline{w_{y\circ}^2}}}.$$

Since $\overline{w_{x\circ}^2}$, $\overline{w_{y\circ}^2}$ and $\overline{w_{z\circ}^2}$ are nearly equal to each other, we may replace $\sqrt{\overline{w_{y\circ}^2}}$ by u', which denotes the root-mean-square value of the velocity fluctuation component at the center of the wake. Then

$$\frac{\lambda}{y_\circ}=\left(\frac{4a}{l}\right)^{1/2}\sqrt{\frac{\nu}{y_\circ u'}}\doteq 1.4\sqrt{\frac{\nu}{y_\circ u'}}. \tag{5.19}$$

This is a relation which was obtained by Taylor[9] based upon entirely different conceptions.

We shall now proceed to evaluate the constants h_i' from the following relations obtained in § 2:

$$a''=-\frac{h_1'}{k_{(11)}\,l-1},\quad a=-\frac{h_3'}{k_{(22)}\,l-1},\quad a'=-\frac{h_5'}{k_{(33)}\,l-1}. \tag{5.20}$$

From the second equation, using the values $k_{(22)}=5$ and $l=0.6$, we have $h_3'=-2a$. Substituting this value into the seventh equation of (2.8), we find

$$\overline{w^3}\sim\frac{U_\circ^2}{x^{\frac{3}{2}}}2a\eta\sim\frac{U_\circ^2}{x^{\frac{3}{2}}}2aD^{3/2}, \tag{5.21}$$

in which we have chosen D, the diameter of the cylinder, as the unit of measurement of length in conformity with (2.14). By means of the relation, $a=x\overline{w_{y\circ}^2}/DU_\circ^2$, we obtain for $x=36D$ in Fage and Falkner's experiment

$$\overline{w^3}/\overline{w^2}\sim|w_y|\sim 2U_\circ/\sqrt{x}=2U_\circ/6=0.3U_\circ, \tag{5.22}$$

which is consistant with the observed result $|w_y|/U_\circ=0.3$. The other constants h_1' and h_5' can be evaluated in a similar way.

It will further be noted that if the assumption $\overline{\omega_{,i}w_k}+\overline{\omega_{,k}w_i}=0$, which we have put forth in the first paragraph of § 2, does not hold (this would mean that the pressure fluctuation may also contribute directly to the production of turbulence), we can still get the same final results by assuming that $\overline{\omega_{,i} w_k} + \overline{\omega_{,k} w_i} = \text{const.}$, since what we have to do then is only to absorb these constants into the functions h_i.

In conclusion, I wish to express my thanks to my teacher Prof. P. Y. Chou for suggesting to me this problem and for his constant guidance during the course of investigation. I also like to express my gratitude to Mr. C. C. Lin for our discussions of the methods of solving the jet problem and the present one.