

THE TURBULENT JET*

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ABSTRACT

The theory of the spread of turbulent jets, based upon Chou's improved general theory of turbulence, is given in the present paper. Starting from the differential equations for mean motion and the equations of double correlations for both velocity and temperature fluctuations, the author has calculated the mean velocity and the mean temperature distributions in the two-dimensional, in the axially symmetrical and in the half jet. In all three cases the same procedure of approximation has been followed. The results are compared with experimental data, and the agreement between theory and experiment is quite good.

1. INTRODUCTION

The theory of the spread of turbulent jets, based upon Chou's general theory of turbulence¹, has been given by Lin². He has calculated the velocity and temperature distributions both in the two-dimensional and in the axially symmetrical jet. The results, except near the edges of the jets, agree quite well with the experiments.

The general theory has been improved recently³. The present paper deals with the steady velocity and temperature distributions in the two-dimensional, axially symmetrical, and half jets, using the

* An improved version of an M. Sc. dissertation, National Tsing Hua University, 1943.

1. P. Y. Chou, *Chinese J. Phys.* 4, 1-33 (1940).
2. C. C. Lin, "Velocity and temperature distributions in turbulent jets", *Science Report of National Tsinghua University*, series A (1941), (printed but failed to appear).
3. P. Y. Chou, "On velocity correlations and solutions of the equations of turbulent fluctuation", *Quarterly J. Appl. Math.* (in press).

improved version of the theory. It is found that Lin's results for the first two kinds of jets still hold true as a first approximation. But the equations which determine the distributions of mean velocity can now be integrated with a better approximation and the theoretical predictions thus obtained agree with the experimental values even up to the edges of the jets.

The half jet has been treated by Tollmien⁴ on the basis of Prandtl's momentum transport theory; theory and experiment agree well in this case. We shall show below that Tollmien's formula of mean velocity distribution also holds good in the present theory.

As in Lin's theory of the spread of turbulent jets we use only the differential equations for mean motion and the equations of double correlation for both velocity and temperature distributions. We also assume as before that in the first two kinds of jets, the magnitudes of the components of the turbulent velocity and temperature fluctuations are constant across the jets, and that the triple velocity correlations are replaced by their respective values in the middle of the jets, while for the half jet we have to use, instead, von Kármán's similarity hypothesis⁵ that the magnitudes of the components of the velocity fluctuation are mutually proportional and that the components of triple correlations are also proportional. In fact this similarity hypothesis when applied to the first two kinds of jets, leads necessarily to the former condition on the double and triple velocity correlations stated before, when the equations of double correlation are put in the approximate form.

PART I. THE TWO-DIMENSIONAL JET

2. EQUATIONS OF MEAN VELOCITY AND TEMPERATURE DISTRIBUTIONS

The equations of mean velocity and temperature distributions have been reduced to simpler forms by Lin. As Lin's paper has not been published, we shall copy his results below with a brief indication of how to obtain them.

4. W. Tollmien, *Zeit. f. angew. Math. u. Mech.* 6, 468-478 (1926).

5. Th. v. Kármán, *Gött. Nachr.* 52-76, (1930).

Let the positive x -axis indicate the direction of mean motion, and let U , V be the components of mean velocity along x and along a perpendicular direction y respectively. If we neglect the viscosity term and apply the boundary layer approximation, which amounts to neglecting $\partial f / \partial x$ in comparison with $\partial f / \partial y$ for any quantity f , the approximate form of the equation of mean motion is

$$(2.1) \quad U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = \frac{1}{\rho} \frac{\partial \tau_{xy}}{\partial y},$$

where $\tau_{xy} = -\rho \overline{w_x w_y}$ is the Reynolds stress, w_x , w_y , w_z being the components of turbulent velocity fluctuation. To solve the above equation Lin assumes the mean velocity distribution of the jet at large distances from the slit to be similar so that every mean quantity is of the form $x^q f(y/x^q)$. (Then substituting into (2.1), applying the condition that the flow of momentum M per unit time across a section of unit height is constant, where

$$M = \rho \int_{-\infty}^{+\infty} U^2 dy$$

and assuming τ_{xy} to vary with x as U^2 , Lin obtains the result, in terms of the stream function ψ ,

$$U = \frac{\partial \psi}{\partial y}, \quad V = -\frac{\partial \psi}{\partial x}, \quad \psi = Ax^{\frac{1}{2}} F(\eta), \quad \left(\eta = \frac{y}{x^{\frac{1}{2}}} \right), \quad (2.2)$$

where A is a constant.

Introducing the functions f_0, f_1, \dots (only f_2 is required for the mean motion; the other functions are useful later on) (Lin (3.10)):

$$\begin{aligned} -p/\rho - \frac{1}{2} A^2 x^{-1} f_0(\eta), \quad -\tau_{xx}/\rho = w_x^2 = \frac{1}{2} A^2 x^{-1} f_1(\eta), \\ -\tau_{xy}/\rho = w_x w_y = \frac{1}{2} A^2 x^{-1} f_2(\eta), \quad (2.3) \\ -\tau_{yy}/\rho = w_y^2 = \frac{1}{2} A^2 x^{-1} f_3(\eta), \quad -\tau_{zz}/\rho = w_z^2 = \frac{1}{2} A^2 x^{-1} f_4(\eta). \end{aligned}$$

We obtain from the equation of motion (2.1) the following equation for F (Lin (3.13))

$$FF' = f_2(\eta) \quad (2.4)$$

By a similar procedure (here the amount of heat carried across a cross-section per unit time is constant) the mean temperature Θ can be put into the form (Lin (3.27))

$$\Theta = Bx^{-\frac{1}{2}} G_1(\eta); \quad (2.5)$$

with B as a constant, and the double velocity and temperature correlations are equal to (Lin (3.29))

$$\overline{w_x \theta} = \frac{1}{2} AB x^{-1} g_1(\eta), \quad \overline{w_y \theta} = \frac{1}{2} AB x^{-1} g_2(\eta). \quad (2.6)$$

Finally the approximate form of the equation for mean temperature distribution can be written as (Lin (3.31))

$$FG = g_2. \quad (2.7)$$

3. EQUATIONS OF DOUBLE CORRELATIONS

(a) Velocity correlation

According to Chou's general theory of turbulence we have the following equations of steady velocity correlation expressed in tensor form:

$$\begin{aligned} (3.1) \quad U_{ij,k} \tau_{ik} + U_{k,j} \tau_{ik} + U_{ik} \tau_{ik} - \rho (\overline{w_j w_i w_k})_{,j} \\ = \rho \left(a_{mik} U^m + b_{ik} \right) \\ + \frac{\lambda^2}{3} \left\{ \frac{2m+1}{3} q^2 g_{ik} + (m-2) S w_{,ik} \right\}, \end{aligned} \quad (3.1)$$

after neglecting the term $U \nabla^2 \tau_{ik}$ which is small at points away from walls. In this set of equations, w represents the amplitude of velocity fluctuation, and λ is Taylor's scale of micro-turbulence. The function S is given by

$$S = - \frac{1}{6 |w_j w_k|} q^2 (q^4 - \rho w^3 w_{,p} w_{,p}), \quad (3.2)$$

where $|\overline{w_i w^k}|$ is the determinant whose (i, k) element is $\overline{w_i w^k}$.

Applying this set of equations to the case of a two-dimensional jet ($x^1 = x$, $x^2 = y$, $x^3 = z$) for which all the average quantities containing an odd number of the index 3 vanish due to symmetry, we have four equations from (3.1) for $(i, k) = (1, 1)$, $(1, 2)$, $(2, 2)$ and $(3, 3)$. The remaining two equations of (3.1) of double velocity correlation, for which $(i, k) = (1, 3)$ and $(2, 3)$, become identities.

Let us again introduce Prandtl's boundary layer approximations and write (Lin (4.11))

$$\begin{aligned}\overline{w_x^2} &= \frac{1}{2} A^2 x^{-\frac{5}{2}} h_0(\eta), & \overline{w_x^2 w_y} &= \frac{1}{2} A^2 x^{-\frac{5}{2}} h_1(\eta), \\ \overline{w_x w_y^2} &= \frac{1}{2} A^2 x^{-\frac{5}{2}} h_2(\eta), \\ \overline{w_y^2} &= \frac{1}{2} A^2 x^{-\frac{3}{2}} h_3(\eta), & \overline{w_x w_z^2} &= \frac{1}{2} A^2 x^{-\frac{5}{2}} h_4(\eta), \\ \overline{w_y w_z^2} &= \frac{1}{2} A^2 x^{-\frac{5}{2}} h_5(\eta);\end{aligned}\tag{3.9}$$

and set furthermore (Lin (4.10))

$$2\nu/\Lambda^2 = A x^{-\frac{5}{2}} l_0(\eta).\tag{3.4}$$

Since components of the tensors a_{mnik} and b_{ik} are either constants or slowly varying functions of the coordinates as proved in the general theory, we put, in accordance with (2.3) and (3.3),

$$a_{mnjk} = \frac{1}{2} A^2 x^{-1} c_{mnik}, \quad b_{ik} = \frac{1}{2} A^2 x^{-1} d_{ik}\tag{3.5}$$

in which c_{mnik} and d_{ik} are now slowly varying functions of η , which may approximately be regarded as constants. On substituting (3.5) into the four non-vanishing equations of (3.1), it is found that some of the c_{mnik} and d_{ik} must be even functions of η and some odd. As in the first approximation we take only the constant term in these slowly varying functions, the odd ones can be put equal to zero. Also it follows from (2.3) that

$$q^2 = \overline{w_x^2} + \overline{w_y^2} + \overline{w_z^2} = \frac{1}{2} A^2 x^{-1} (f_1 + f_2 + f_3).\tag{3.6}$$

Substituting these relations into the four equations derived from (3.1) and considering that those components of a_{mmik} and b_{ik} which are odd functions of η vanish, we have

$$2(F'' + \eta F''')f_1 - 2F''f_2 + \frac{1}{2}h_0 + \frac{1}{2}Ff_1' + \eta h_0' = h_1' \\ = \frac{1}{2}(c_{2211} - c_{1111})(F'' + 2\eta F''') + d_{11} \\ + \frac{1}{3}(2m+1)I_0(f_1 + f_2 + f_4) + (m-2)S_0 f_4 \quad (3.7)$$

$$\eta(\eta^2 F'' + \eta F' - \frac{1}{2}F)f_1 + Ff_2' - F''f_3 + \frac{1}{2}h_1 + \frac{1}{2}Ff_2' + \eta h_1' = h_2' \\ = c_{2112}F'' - c_{1212}(\eta^2 F'' + \eta F' - \frac{1}{2}F) + (m-2)S_0 f_2 \quad (3.8)$$

$$2(\eta^2 F'' + \eta F' - \frac{1}{2}F)f_2 - 2\eta F''f_3 + \frac{1}{2}h_2 + \frac{1}{2}Ff_3' + \eta h_2' = h_3' \\ = \frac{1}{2}(c_{2222} - c_{1122})(h_2 + 2\eta F'') + d_{22} \\ + \frac{1}{3}(2m+1)I_0(f_1 + f_2 + f_4) + (m-2)S_0 f_4 \quad (3.9)$$

$$F'f_4 + \frac{1}{2}h_4 + \frac{1}{2}Ff_4' + \eta h_4' = h_5' \\ = \frac{1}{2}(c_{2233} - c_{1133})(F' + 2\eta F'') + d_{33} \\ + \frac{1}{3}(2m+1)I_0(f_1 + f_2 + f_4) + (m-2)S_0 f_4 \quad (3.10)$$

Following Lin, let us compare the order of magnitude of the different terms in the above set of equations. After neglecting small quantities of orders higher than η_0 which is defined by* (Lin (4.17)),

$$E(\eta_0) = \frac{1}{2}, \quad (3.11)$$

and assuming the even function h to be constant, we find that (3.8) reduces to the form

$$F'' + (a - \eta F')f_2 = \frac{1}{2}Ff_2' \quad (3.12)$$

where $a = (m-2)S_0$ and $\delta = c_{2112}/F$ may be regarded as constants. This equation is substantially the same as the equation (4.21) of Lin obtained on the old theory. In actual calculation Lin replaces this equation by the approximate one (Lin (4.22));

* The experimental value of η_0 is about 0.7.

$$-bF'' + af_2 = 0. \quad (3.12a)$$

We shall, however, use the more accurate equation (3.12).

(b) *Temperature correlation*

The tensor equation of steady double temperature correlation runs as

$$U^j (w_i \theta)_{,j} + w_i w^j \theta_{,ij} + w^j \theta U_{i,j} + (w^j w_i \theta)_{,j} \\ = - \frac{n}{mi} U_{,i}^m - b_i - \frac{K+\nu}{\Lambda^2} (3 B_{ip} w^p \theta - B_0 w_i \theta) \quad (3.13)$$

where B_{ip} can be taken to be constants, since both θ_1 , which denotes the amplitude of temperature fluctuation, and q are approximately constant, and $B_0 = B_{ij}$. In rectangular coordinates it leads to two scalar equations for $i=1$ and 2. The third component corresponding to $i=3$ vanishes identically.

Considering the even and odd properties of the various functions, we have the constants $B_{12} = B_{21} = 0$, and functions $a_{211} = a_{121} = a_{112} = a_{222} = b_2 = 0$. Let us write, as an analogue to (3.3) and (3.4),

$$(w_x^2 \theta = \frac{1}{2} A^2 B x^{-\frac{3}{2}} k_0(\eta), \quad w_x w_y \theta = \frac{1}{2} A^2 B x^{-\frac{3}{2}} k_1(\eta), \\ w_y^2 \theta = \frac{1}{2} A^2 B x^{-\frac{3}{2}} k_2(\eta), \quad (3.14)$$

$$(K+\nu)/\Lambda^2 = A x^{\frac{3}{2}} L_0(\eta), \quad (3.15)$$

and put, as in (3.5) and (3.6),

$$a_{nmi} = \frac{1}{2} A B x^{-1} c_{nmi}, \quad b_i = \frac{1}{2} A^2 B x^{-\frac{5}{2}} d_i. \quad (3.16)$$

We obtain from the two equations of velocity-temperature correlation the following two equations, after neglecting small quantities of orders higher than unity,

$$f_2 G' + g_2 F'' + k_1' = \frac{1}{2} (c_{111} - c_{221}) (F' + 2\eta F'') - d_1 \\ - (3B_{11} - B_0) L_0 \theta_1, \quad (3.17)$$

$$f_2 G' + k_2' = -c_{2,2} F'' - (3B_{2,2} - B_0) L_0 g_2. \quad (3.18)$$

The first equation gives g_1 , and the second determines g_2 . Let us now consider the second equation only. We assume, analogously to the theory of velocity fluctuation, k_2 and L_2 to be constant. Then (3.18) becomes

$$G' = \alpha F'' + \beta g_2, \quad (3.19)$$

with the constants $\alpha = -c_{2,2}/f_2$ and $\beta = -L_0(3B_{2,2} - B_0)$. This reduces to Lin's equation (4.43) when the term $\alpha F''$ is neglected.

4. DISTRIBUTION OF MEAN VELOCITY AND TEMPERATURE

(a) Velocity distribution

Eliminating f_2 from (2.4) and (3.12) we obtain

$$(b - \frac{1}{2}F^2)F'' + aFF' - \frac{1}{2}FF'^2 = 0, \quad (4.1)$$

The solution of this equation satisfying the condition $F=0$, $F'=1$ at $\eta=0$ is*

$$\left. \begin{aligned} \frac{U}{U_m} = F' &= \frac{1}{1-\sigma^2} \left\{ 1 - \sigma^2 \left(\frac{1+\xi^2}{1-\xi^2} \right)^2 \right\}, \\ \frac{\eta}{\sqrt{2b}} &= (1-\sigma^2) \frac{2\xi}{1+\xi^2} + \frac{1}{2}\sigma^2(1-\sigma^2)I, \end{aligned} \right\} \quad (4.2)$$

where $U_m = Ax^{-\frac{1}{2}}$ is the maximum value of U and σ is a constant connected with a by the relation

$$1 - \sigma^2 = 3/2a.$$

The relation between F and ξ is

$$\frac{F}{\sqrt{2b}} = \frac{2\xi}{1+\xi^2}.$$

* I wish to thank Prof. J. S. Wang for pointing out this solution to me and for many valuable suggestions which enable me to bring the paper to the present form.

and l is a function of ξ given by

$$I = \frac{1}{\sqrt{1-\sigma^2}} \log \frac{\gamma + \xi}{\gamma - \xi} - \frac{(1+\sigma)(\alpha^2 + \beta^2 + \gamma^2)}{4\alpha\sqrt{1+\sigma^2+\sigma^4}} \log \frac{(\alpha + \xi)^2 + \beta^2}{(\alpha - \xi)^2 + \beta^2} \\ + \frac{(1+\sigma)(\alpha^2 + \beta^2 - \gamma^2)}{2\beta\sqrt{1+\sigma^2+\sigma^4}} \left(\tan^{-1} \frac{\alpha + \xi}{\beta} - \tan^{-1} \frac{\alpha - \xi}{\beta} \right),$$

where α, β, γ are constants defined by

$$\alpha^2 + \beta^2 = \frac{\sigma\sqrt{1+\sigma^2+\sigma^4}}{1+\sigma+\sigma^2}, \quad \alpha^2 - \beta^2 = \frac{1-\sigma^2}{1-\sigma+\sigma^2}, \quad \gamma^2 = \frac{1-\sigma}{1+\sigma}.$$

To compare the solution (4.2) with experiment, we assume a value of σ and calculate the corresponding velocity distribution. By trial it is found that $\sigma = 0.9$ ($a = 5.53$) gives the best agreement with experiment. The theoretical result is compared with Förlthmann's experiments⁶ (represented by \circ and \otimes) in fig. 1. The calculated value of η_0 (at which $\bar{u}' = \frac{1}{2}$) is $0.404\sqrt{2b}$, which gives $b \sim 0.0307$ for $\eta_0 \sim 0.1$.

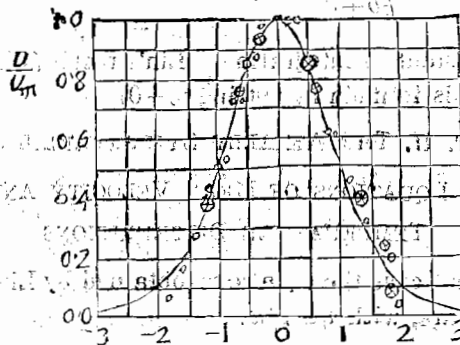


Fig. (1) — Velocity distribution in a plane jet

6. K. Förlthmann, *Ing.-Arch.* 5, 42-54 (1934).

(b) Temperature distribution

Eliminating g , from (2.7) and (3.9) we obtain

$$G' = \alpha F'' + \beta FG, \quad (4.3)$$

whose solution is

$$G = \alpha F' + \alpha\beta e^{\beta \int F d\eta} \int e^{-\beta \int F d\eta} F d\eta. \quad (4.4)$$

The integral $\int F d\eta$ can be evaluated in finite terms from the solution (4.2), but the last integral in (4.4) should be evaluated by numerical integration. Since no experimental data are available, the numerical integration has not been carried out.

If we use the approximate equation (3.12a), then (4.3) reduces to

$$G' + \frac{\beta b}{a} \frac{F''}{F'} G = \alpha F'', \quad (4.5)$$

with the solution

$$G = \frac{\alpha a}{\beta b + a} F' + C (F')^{-\beta b/a} \quad (4.6)$$

where C is a constant of integration. Lin's result (Lin (5.4)) can be obtained from this formula by putting $\alpha = 0$.

PART. II. THE AXIALLY-SYMMETRICAL JET

5. EQUATIONS OF MEAN VELOCITY AND TEMPERATURE DISTRIBUTIONS

In the present case the equations obtained by Lin corresponding to (2.2)–(2.7) are, with $\eta = r/x$,

$$U = Ax^{-1} F'(\eta)/\eta, \quad V = Ax^{-1} \{ \eta F'(\eta) - F(\eta) \} / \eta, \quad (5.1)$$

$$\begin{aligned} p &= -\rho A^2 x^{-2} f_0(\eta), & \tau_{xx} &= -\rho A^2 x^{-2} f_1(\eta), \\ \tau_{xr} &= -\rho A^2 x^{-2} f_2(\eta), \\ \tau_{rr} &= -\rho A^2 x^{-2} f_3(\eta), & \tau_{\theta\theta} &= -\rho A^2 x^{-2} f_4(\eta), \end{aligned} \quad (5.2)$$

$$FF' = \eta^2 f_2, \quad (5.3)$$

$$\Theta = Bx^{-1}G(\eta), \quad (5.4)$$

$$\overline{w_x \theta} = ABx^{-2}g_1(\eta), \quad \overline{w_r \theta} = ABx^{-2}g_2(\eta), \quad (5.5)$$

$$FG = -\eta g_2. \quad (5.6)$$

6. EQUATIONS OF DOUBLE CORRELATIONS

(a) Velocity correlation

When (3.1) is expressed in polar coordinates ($x^1 = x$, $x^2 = r$, $x^3 = \theta$), four equations are obtained, which need not be written down explicitly. Let us put, as in the former case,

$$\left. \begin{aligned} \overline{w_x^2} &= A^2 x^{-2} h_0(\eta), & \overline{w_x w_r} &= A^2 x^{-2} h_1(\eta), \\ \overline{w_x w_r^2} &= A^2 x^{-2} h_2(\eta), & \overline{w_r^2} &= A^2 x^{-2} h_3(\eta), \\ \overline{w_x w_\theta^2} &= A^2 x^{-2} h_4(\eta), & \overline{w_r w_\theta^2} &= A^2 x^{-2} h_5(\eta), \end{aligned} \right\} \quad (6.1)$$

$$2\nu/\Lambda^2 = A^2 x^{-2} l_0(\eta), \quad (6.2)$$

$$a_{mnk} = A^2 x^{-2} c_{mnk}, \quad b_{ik} = A^2 x^{-2} d_{ik}. \quad (6.3)$$

We can still use the values of the double and triple velocity correlations in the central portion of the jet as their corresponding values for the whole jet as a first approximation. Thus functions $f_0, f_1, f_2, f_4, h_0, h_2$ and h_4 will be assumed to be constants, while h_1, h_3 and h_5 will be proportional to η as before.

After substituting expressions from (6.1)–(6.3) into the four equations of double correlation and considering the values of the different terms in the second of these four equations at $\eta = 0$ (notice that $F'/\eta = 1$ at $\eta = 0$), we find that if we neglect terms of order η^2 , we can put $c_{1112}, c_{2212}, c_{3312}$ and d_{12} to be zero, together with

$$h_2 = h_4 \quad \text{or} \quad \overline{w_x(w_r^2 - w_\theta^2)} = 0. \quad (6.4)$$

The resulting equations of double correlation involving the non-vanishing terms become

$$\begin{aligned}
 2(F'' + \frac{F'}{\eta})f_1 - 2(\frac{F''}{\eta} - \frac{F'}{\eta^2})f_2 + 3h_0 - \frac{h_1}{\eta} - h_1' \\
 = -c_{1111}F'' + c_{2211}(F'' - \frac{F'}{\eta} + \frac{F'}{\eta^2}) + c_{3311}(\frac{F'}{\eta} - \frac{F'}{\eta^2}) + d_{11} \\
 + \frac{2m+1}{3}l_0(f_1 + f_3 + f_4) + (m-2)Sl_0f_1, \quad (6.5)
 \end{aligned}$$

$$\begin{aligned}
 \eta F''f_1 + (\frac{3F'}{\eta} - \frac{F'}{\eta^2})f_2 - (\frac{F''}{\eta} - \frac{F'}{\eta^2})f_3 + \varepsilon h_1 + \frac{F'}{\eta}f_2' + \eta h_1' \\
 = c_{3112}(\frac{F''}{\eta} - \frac{F'}{\eta^2}) - c_{1212}\eta F'' + (m-2)Sl_0f_2, \quad (6.6)
 \end{aligned}$$

$$\begin{aligned}
 2\eta F''f_2 - 2(F'' - \frac{2F'}{\eta} + \frac{F'}{\eta^2})f_3 + \varepsilon h_2 + \frac{1}{\eta}(2h_5 - h_5') - h_5' \\
 = -c_{1122}F'' + c_{2222}(F'' - \frac{F'}{\eta} + \frac{F'}{\eta^2}) + c_{3322}(\frac{F'}{\eta} - \frac{F'}{\eta^2}) + d_{22} \\
 + \frac{2m+1}{3}l_0(f_1 + f_3 + f_4) + (m-2)Sl_0f_2, \quad (6.7)
 \end{aligned}$$

$$\begin{aligned}
 \frac{2F'}{\eta^2}f_4 + 3h_4 - \frac{3h_5}{\eta} - h_5' \\
 = -c_{1133}F'' + c_{2233}(F'' - \frac{F'}{\eta} + \frac{F'}{\eta^2}) + c_{3333}(\frac{F'}{\eta} - \frac{F'}{\eta^2}) + d_{33} \\
 + \frac{2m+1}{3}l_0(f_1 + f_3 + f_4) + (m-2)Sl_0f_4, \quad (6.8)
 \end{aligned}$$

If small quantities of orders higher than η_0 are neglected (η_0 is the root of $F'/\eta = \frac{1}{2}$ and is approximately equal to 0.1), (6.6) becomes

$$(c_{2,1,1} + f_2) \frac{d}{d\eta} \left(\frac{F'}{\eta} \right) + [(m-2)St_0 - \frac{3F'}{\eta} + \frac{F}{\eta^2}] f_2 = \frac{F}{\eta} f_2' \quad (6.9)$$

(b) *Temperature correlation*

Analogously to (3.14)-(3.18) we have

$$\left. \begin{aligned} \overline{w_x^2 \theta} &= A^2 B x^{-3} k_0(\eta), & \overline{w_x w_r \theta} &= A^2 B x^{-3} k_1(\eta) \\ \overline{w_r^2 \theta} &= A^2 B x^{-3} k_2(\eta), & \overline{w_\theta^2 \theta} &= A^2 B x^{-3} k_3(\eta) \end{aligned} \right\} \quad (6.10)$$

$$(K + \nu) / \Lambda^2 = A x^{-2} D_0(\eta), \quad (6.11)$$

$$a_{nmi} = A B x^{-2} c_{nmi}, \quad b_i = A^2 B x^{-2} d_i, \quad (6.12)$$

$$\begin{aligned} f_2 G' + g_2 \frac{d}{d\eta} \left(\frac{F'}{\eta} \right) + \frac{k_1}{\eta} + k_1' &= -c_{2,1,1} \frac{d}{d\eta} \left(\frac{F'}{\eta} \right) - d_1 \\ &\quad - L_0 \left\{ (3B_{1,1} - B_0) g_1 + 3B_{1,2} g_2 \right\}, \end{aligned} \quad (6.13)$$

$$\begin{aligned} f_2 G' + \frac{1}{\eta} (k_2 - k_1) + k_2' &= -c_{2,2,2} \frac{d}{d\eta} \left(\frac{F'}{\eta} \right) - d_2 \\ &\quad - L_0 \left\{ (3B_{2,1} g_1 + (3B_{2,2} - B_0) g_2 \right\}, \end{aligned} \quad (6.14)$$

In the above equations we can also apply the condition at $\eta=0$ to eliminate some of the constants c_{nmi} , d_i and B_{ik} . Furthermore k_1 is assumed to be proportional to η while k_2 is constant. This gives $d_2=0$; $B_{2,1}=0$, and

$$k_2 = k_3 \quad \text{or} \quad (\overline{w_r^2} - \overline{w_\theta^2}) \theta = 0. \quad (6.15)$$

The approximate form of the equation (6.14) is

$$(S7) \quad G' = \alpha \frac{d}{d\eta} \left(\frac{F'}{\eta} \right) + \beta g_2, \quad (6.16)$$

where the constants α and β are also given by (3.19).

7. DISTRIBUTIONS OF MEAN VELOCITY AND TEMPERATURE

(a) *Velocity distribution*

Eliminating f_2 from (5.3) and (6.9) we obtain

$$\left(a - \frac{4F'}{\eta} + \frac{2F}{\eta^2}\right) \frac{FF'}{\eta^2} + \left(b - \frac{F^2}{\eta^2}\right) \frac{d}{d\eta} \left(\frac{F'}{\eta}\right) = 0. \quad (7.1)$$

Here the constants a and b are given by the same expressions as those in (3.12). The result of numerical integration of this equation is shown in fig. 2, and compared with the experiments of Ruden⁷ and Kuethe⁸. The value of the constant a which gives the best fit with experiment is 8, and the corresponding value of η_0 is $0.73\sqrt{b}$. Since η_0 is about 0.1, b will be about 0.019.

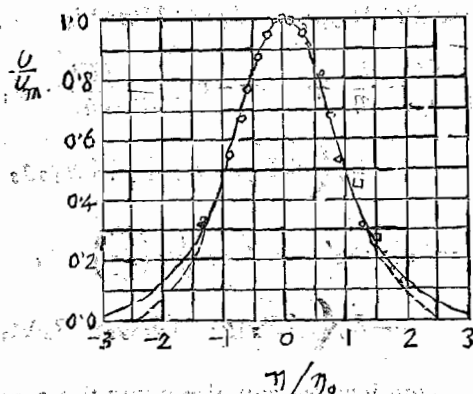


Fig. 2 — Velocity distribution in an axially symmetrical jet

□ — Kuethe's experiment
○ — Ruden's experiment

(b) Temperature distribution

From (5.6) and (6.16) we obtain

$$G' = \alpha \frac{d}{d\eta} \left(\frac{F'}{\eta} \right) - \frac{8F}{\eta} G. \quad (7.2)$$

7. Ruden, *Naturwiss.* 21, 375-378, (1933).

8. Kuethe, *Jour. of Appl. Mech.* 2, 87-95, (1925).

This equation is integrated numerically by assuming suitable values of the constant βb and determining the corresponding values of α by the experimental value of η/η_0 at which $G = \frac{1}{2}$ (Ruden's experiment gives $\eta/\eta_0 = 1.15$, η_0 being the value of η at which $F'(\eta) = \frac{1}{2}$). Complete agreement with experiment cannot be obtained, and the value $\beta b = 10$ is the best that can be chosen (with the corresponding $\alpha = -0.85$). The result is plotted in fig. 3 and compared with Ruden's experiment.

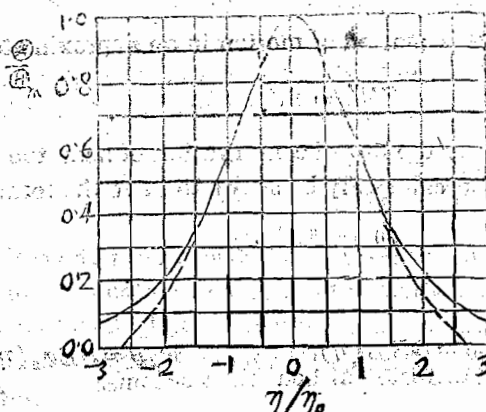


Fig. 3 — Temperature distribution in an axially symmetrical jet
— — Ruden's experiment

PART III. THE HALF JET

8. EQUATIONS OF MEAN VELOCITY AND

TEMPERATURE DISTRIBUTION

A stream of fluid, flowing parallel to the positive x -axis with velocity U_0 , passes over and mixes with the still fluid situated on the side $x > 0$ and at the same level with the solid base to the side $x < 0$. It is required to find the mean velocity distribution in the complete

fluid. By the law of conservation of momentum* and the assumption that $\overline{w_x w_y}$ varies with x according to U^2 , we can deduce the form of the stream function in the present case as

$$\psi = Ax F(\eta) \quad (\eta = y/x) \quad (8.1)$$

Putting

$$\left. \begin{aligned} \overline{p} &= -\rho A^2 f_0(\eta), \quad \tau_{xx} = -\rho A^2 f_1(\eta), \quad \tau_{xy} = -\rho A^2 f_2(\eta), \\ \tau_{yy} &= -\rho A^2 f_3(\eta), \quad \tau_{xz} = -\rho A^2 f_4(\eta) \end{aligned} \right\} \quad (8.2)$$

we arrive at the equation of motion in an approximate form

$$FF'' = f_2' \quad (8.3)$$

Similarly, by the law of conservation of heat the mean temperature distribution can easily be shown to have the form

$$\Theta = BG(\eta) \quad (8.4)$$

Write

$$\overline{w_x \theta} = AB g_1(\eta), \quad \overline{w_y \theta} = AB g_2(\eta), \quad (8.5)$$

the equation of diffusion of heat then becomes

$$FG' = g_2' \quad (8.6)$$

9. EQUATIONS OF DOUBLE CORRELATIONS

(a) Velocity correlation

The equations of double velocity correlation in the present case are identical with those in the two-dimensional jet. Similar to (3.3)-(3.5) we also have

* The flow of momentum per unit time across a section of unit height is

$$M = \rho \int_{y_2}^{y_1} U^2 dy = \rho U_0^2 y_1$$

where y_1 and y_2 are the upper and lower boundaries of the jet.

$$\left. \begin{aligned} \overline{w_x^2} &= A^2 h_0(\eta), & \overline{w_x^2 w_y} &= A^2 h_1(\eta), & \overline{w_x w_y^2} &= A^2 h_2(\eta), \\ \overline{w_y^2} &= A^2 h_3(\eta), & \overline{w_x w_z^2} &= A^2 h_4(\eta), & \overline{w_y w_z^2} &= A^2 h_5(\eta), \end{aligned} \right\} \quad (9.1)$$

$$2\nu/\lambda^2 = Ax^{-1}I_0(\eta), \quad (9.2)$$

$$a_{mnik} = A^2 c_{mnik}, \quad b_{ik} = A^2 x^{-1} d_{ik} \quad (9.3)$$

In the plane and axial jets we have taken the values of f_1, f_2 and f_4 and h_1, h_2, h_3, h_4 and h_5 in the central portion of the jets. This proves to be ineffective in the present case and we have to adopt von Kármán's similarity hypothesis⁶ which can be shown to lead to the former procedure for the first two kinds of jets, if terms of order higher than η_0 are neglected. This hypothesis can be expressed as

$$\left(\frac{f_1}{r_1} = \frac{f_2}{r_2} = \frac{f_4}{r_4} = f \quad \text{and} \quad \frac{h_1}{t_1} = \frac{h_2}{t_2} = \frac{h_3}{t_3} = \frac{h_4}{t_4} = \frac{h_5}{t_5} = h, \right) \quad (9.4)$$

where the symbols r and t are constants. By this assumption S may still be regarded approximately as a constant. With the aid of (8.1), (8.2) and (9.1)-(9.4) and after neglecting small quantities of orders higher than η_0 , the four equations of double correlations become in the first approximation

$$-2f_2 F'' - t_1 h' = c_{2111} F'' + d_{11} + I_0 \left\{ \frac{1}{2}(2m+1) Rf + (m-2) S r_1 f \right\}, \quad (9.5)$$

$$-r_3 f F'' - t_2 h' = c_{2112} F'' + d_{12} + I_0 (m-2) S f_2, \quad (9.6)$$

$$-t_3 h' = c_{2122} F'' + d_{22} + I_0 \left\{ \frac{1}{2}(2m+1) Rf + (m-2) S r_2 f \right\}, \quad (9.7)$$

$$-t_5 h' = c_{2133} F'' + d_{33} + I_0 \left\{ \frac{1}{2}(2m+1) Rf + (m-2) S r_4 f \right\}, \quad (9.8)$$

in which $R = r_1 + r_2 + r_4$ is a constant.

If the orders of magnitude of f_1, f_2 and $f_4; h_1, h_2, h_3$ and $h_5; c_{2111}, c_{2122}$ and c_{2133} and d_{11}, d_{22} and d_{33} are respectively the same but different for the different kinds of functions, then the term $-2f_2 F''$ can be neglected by comparing (9.5) with (9.6)-(9.8). Now absorb the constants d into the functions h' , and assume I_0 to be a constant;

then (9.5)-(9.8) give consistently that

$$h' = e_1 F'', \quad f = e_2 F'', \quad (9.9)$$

and

$$f_2 = e F''', \quad (9.10)$$

where e_1 , e_2 and e are constants.

(b) Temperature correlation

The equations of double correlation for heat transfer are the same as in the case of the plane jet. For the triple correlations we write analogously

$$\overline{w_x^2 \theta} = A^2 B k_0(\eta), \quad \overline{w_x w_y \theta} = A^2 B k_1(\eta), \quad \overline{w_y^2 \theta} = A^2 B k_2(\eta), \quad (9.11)$$

$$(K + \nu) / \Lambda^2 = A x^{-1} L_0(\eta), \quad (9.12)$$

$$c_{nmi} = AB c_{nmi}, \quad b_i = A^2 B x^{-1} d_i. \quad (9.13)$$

We shall use the similarity hypothesis again as in (9.4) by putting

$$g_1 / s_1 = g_2 / s_2 = g, \quad (9.14)$$

where s_1 and s_2 are two constants. Substituting these relations into the two equations of temperature correlation and retaining quantities of the largest magnitude in these equations thus obtained, we have

$$f_2 G' + g_2 F'' = -c_{211} F'' - \{ (3B_{11} - B_0) s_1 + 3B_{12} s_2 \} L_0 g. \quad (9.15)$$

$$f_1 G' = -c_{212} F'' - \{ 3B_{21} s_1 + (3B_{22} - B_0) s_2 \} L_0 g. \quad (9.16)$$

From these equations together with (9.9) and (9.10), it follows that

$$g_2 = A + G' (B - e F''), \quad (9.17)$$

where A and B are two constants.

10. DISTRIBUTIONS OF MEAN VELOCITY AND TEMPERATURE

(a) Velocity distribution

It follows from (8.3) and (9.10) that

$$FF'' - 2eF''F''' = 0. \quad (10.1)$$

This equation is of the same form as that derived by Tollmien on the basis of the momentum transport theory. The theoretical result agrees well with experiment.

(b) *Temperature distribution*

From (8.6), (9.17) and (10.1) we have

$$G' = C(eF'' - B)^{1/2}, \quad (10.2)$$

where C is a constant of integration. No experimental data for this case are available.

I wish to express my sincere thanks to my teacher Professor P. Y. Chou for suggesting this problem and also for his guidance during the course of investigation.