THE TURBULENT FLOW THROUGH A CIRCULAR PIPE*

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ABSTRACT

Chou's general theory of turbulence is applied to the turbulent flow through a circular pipe. The method of solution employed by Chou in his investigation of the turbulent flow through a channel is used here and leads to results which agree well with experiment.

1. Introduction

Based upon Chou's general theory of turbulence[1] C.C. Lin[2] has solved the problem of pressure flow of a turbulent fluid through a circular pipe. His results agree very well with the existing experimental data. The general theory has been improved in the mean time.[3] The present solution of the problem is based upon the improved version of the theory. The excellent agreement between the experimental measurements of the mean velocity distribution and of the magnitude of components of velocity fluctuation with theory can be considered as a strong support of the present new point of view.

For the sake of convenience we list below the various equations of motion which have been derived in Chou's papers:

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1. P. Y. Chou, *Chinese J. Phys.* 4,1 (1940).
2. C.C. Lin "Pressure Flow of a Turbulent Fluid through a Circular Pipe" (unpublished).
3. P. Y. Chou, "On Velocity Correlations and the Solutions of the Equations of Turbulent Fluctuation" (to be published).

$$\frac{\partial U_i}{\partial t} + U^j U_{i,j} = -\frac{1}{\rho}\overline{P}_{,i} + \frac{1}{\rho}\tau^j_{i,j} + \nu\nabla^2 U_i, \qquad U^j_{,j} = 0 \quad (1.1)$$

$$\frac{1}{\rho}\frac{\partial}{\partial t}\tau_{ik} - \frac{1}{\rho}\left(U_{i,j}\,\tau^j_k + U_{k,j}\,\tau^j_i\right) - \frac{1}{\rho}U^j\tau_{ik,j} + \left(\overline{w^j w_i w_k}\right)_{,j}$$

$$= -a^n_{mik}U^m_{,n} - b_{ik} - \frac{2\nu}{\lambda^2}\left[\frac{1}{3}\,(2m+1)\,q^2 g_{ik} + (m-2)\,S\,\overline{w_i w_k}\right], \qquad (1.2)$$

$$\frac{\partial}{\partial t}(\overline{w_i w_k w_l}) + U_{i,j}\overline{w^j w_k w_l} + U_{l,j}\overline{w^j w_i w_k} + U_{k,j}\overline{w^j w_i w_l} + U^j(\overline{w_i w_k w_l})_{,j} + (\overline{w^j w_i w_k w_l})_{,j} = -b^n_{mikl}U^m_{,n} - c_{ikl}$$

$$+\frac{1}{\rho^2}\left(\tau^j_{i,j}\,\tau_{kl} + \tau^j_{k,j}\,\tau_{il} + \tau^j_{l,j}\,\tau_{ik}\right), \quad (1.3)$$

where $S = \dfrac{q^2}{8\,|\overline{w_i w_k}|}\left(q^4 - \overline{w_p w_s}\,\overline{w^p w^s}\right)$, $|\overline{w_i w_k}|$ denoting the determinant whose (ik) element is $\overline{w_i w^k}$.

2. Mean Motion.

In the present calculation cylindrical coordinates are used, in which the z-axis $(z = x^1)$ is chosen as the central axis, $r\,(r = x^2)$ is the radial coordinate, and θ $(\theta = x^3)$ is the azimuthal angle. Since this problem possesses axial symmetry, all the average quantites do not depend upon θ. Moreover the flow is steady and is independent of the position of the cross-section of the pipe; all the average quantities except the mean pressure $\overline{p}$ are functions of r alone.

Let us study the pressure $\overline{p}$ first. Since $\partial\overline{p}/\partial z$ is the same for all the cross-sections, we have $\partial^2 \bar{p}/\partial z^2 = 0$. Similiarly $\partial \bar{p}/\partial r$ is independent of z, and we have,

$$\frac{\partial}{\partial r}\left(\frac{\partial \bar{p}}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial \bar{p}}{\partial r}\right) = 0.$$

This means that $\partial \bar{p}/\partial z$ is the same for all points in a cross section and hence is a constant for all values of r and z.

We define the frictional velocity U_τ by the equation,

$$-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z} = \frac{2}{a} U_\tau^2, \tag{2.1}$$

where a is the radius of the pipe.

Among the components of the mean velocity, U_θ vanishes from symmetry consideration, U_z and U_r are functions of r. The equation of continuity for the mean velocity then becomes

$$\frac{1}{r}\frac{d}{dr}\left(rU_r \right) = 0$$

This gives $rU_r =$ constant, and consequently U_r is equal to zero, for the constant is zero at $r = 0$.

With the above simplifications the three equations of mean motion can be put into the following forms:

$$0 = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z} + \frac{1}{\rho r}\frac{d}{dr}(r\tau_{zr}) + \frac{\nu}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right), \tag{2.2}$$

$$0 = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial r} + \frac{1}{\rho r}\frac{d}{dr}(r\tau_{rr}) - \frac{1}{\rho r}\tau_{\theta\theta}, \tag{2.3}$$

$$0 = \frac{1}{\rho r}\frac{d}{dr}(r^2 \tau_{r\theta}). \tag{2.4}$$

Equation (2.4) gives $r^2 \tau_{r\theta} =$ const. Since $\tau_{r\theta}$ cannot become infinite at $r = 0$, the constant must be zero and we have

$$\tau_{r\theta} = 0. \tag{2.5}$$

From (2.2) we have

$$\tau_{zr} + \mu \frac{dU}{dr} = \frac{r}{2} \frac{\partial p}{\partial z}. \tag{2.6}$$

The additive constant of integration is chosen to be zero in order that τ_{zr} shall be finite at $r = 0$. Since the turbulent stress is very much larger than the laminar stress $\mu\, dU/dr$, the above expression for τ_{zr} can be put into the simpler form,

$$\tau_{zr} = -\frac{r}{a} \rho U_\tau^2. \tag{2.7}$$

By symmetry we have

$$\tau_{z\theta} = \tau_{\theta z} = 0. \tag{2.8}$$

3. Equations of Double and Triple Correlations

In equations (1.2) and (1.3), a^n_{mik}, b_{ik}, b^n_{mikl}, and c_{ikl} are all slowly varying functions, and can be approximated with sufficient accuracy by the first term in their expressions in powers of r. For the explanation of this point the reader is referred to Chou's papers.[3,4] We approximate each one of them by a constant or a constant multiple of σ (where $\sigma = r/a$), according to whether it is even or odd in r. These approximate expressions are listed below:

$$a_{2111} = a_1 U_\tau^2 \sigma \quad b_{21111} = 3\beta_1 U_\tau^3 \sigma \quad b_{11} = b_1 U_\tau^3/a \quad c_{111} = 3c_1 U_\tau^4/a$$

$$a_{2122} = a_2 U_\tau^2 \sigma \quad b_{21122} = \beta_2 U_\tau^3 \sigma \quad b_{22} = b_2 U_\tau^3/a \quad c_{122} = c_2 U_\tau^4/a$$

$$a_{2133} = a_3 U_\tau^2 \sigma^3 \quad b_{21133} = \beta_3 U_\tau^3 \sigma^3 \quad b_{33} = b_3 U_\tau^3 \sigma^2/a \quad c_{133} = c_3 U_\tau^4 \sigma^2/a$$

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4. P. Y. Chou, "Pressure Flow of a Turbulent Fluid between Two Parallel Infinite Planes" (to be published).

$$a_{2112}=a_4U_\tau^2 \qquad b_{21112}=2\beta_4U_\tau^2 \qquad b_{12}=b_4U_\tau^2\sigma/a \quad c_{112}=c_4U_\tau^4\sigma/a$$

By the aid of the above expressions the set of equations embodied in (1.2) become

$$2U_\tau^2\ \sigma\frac{dU}{d\sigma}+\frac{1}{\sigma}\frac{d}{d\sigma}\left(\sigma\,\overline{w_z^2w_r}\right)=-a_1U_\tau^2\sigma\frac{dU}{d\sigma}-b_1U_\tau^2$$
$$-\frac{2\nu\sigma}{\lambda^2}\left[\tfrac{1}{3}(2m+1)q^2+(m-2)\,S\overline{w_z^2}\right], \qquad (3.1)$$

$$\overline{w_r^2}\frac{dU}{d\sigma}+\frac{1}{\sigma}\frac{d}{d\sigma}\left(\sigma\,\overline{w_zw_r^2}\right)-\frac{1}{\sigma}\overline{w_zw_\theta^2}=-a_4U_\tau^2\frac{dU}{d\sigma}$$
$$-b_4U_\tau^2\sigma-\frac{2\nu a}{\lambda^2}(m-2)\,U_\tau^2\sigma S, \qquad (3.2)$$

$$\frac{1}{\sigma}\frac{d}{d\sigma}\left(\sigma\,\overline{w_r^3}\right)-\frac{2}{\sigma}\left(\overline{w_rw_\theta^2}\right)=-a_2U_\tau^2\ \sigma\frac{dU}{d\sigma}-b_2U_\tau^2$$
$$-\frac{2\nu a}{\lambda^2}\left[\tfrac{1}{3}(2m+1)q^2+(m-2)\,S\overline{w_r^2}\right], \qquad (3.3)$$

$$\frac{1}{\sigma}\frac{d}{d\sigma}\left(\sigma^3\overline{w_rw_\theta^2}\right)=-a_3U_\tau^2\sigma^3\frac{dU}{d\sigma}-b_3U_\tau^2\sigma^2$$
$$-\frac{2\nu a}{\lambda^2}\left[\tfrac{1}{3}(2m+1)q^2\sigma^2+(m-2)\,S\sigma^2\overline{w_\theta^2}\right], \qquad (3.4)$$

The equations of (1.3) which do not vanish identically are

$$\overline{w_rw_z^2}\frac{dU}{d\sigma}=-\beta_1U_\tau^2\sigma\frac{dU}{d\sigma}-c_1U_\tau^4-\tfrac{1}{3}\frac{1}{\sigma}\frac{d}{d\sigma}\left(\sigma\,\overline{w_rw_z^2}\right)$$
$$+2U_\tau^2\,\overline{w_z^2}, \qquad (3.5)$$

$$\overline{w_zw_r^2}\frac{dU}{d\sigma}=-\beta_4U_\tau^2\frac{dU}{d\sigma}-c_4U_\tau^4\sigma-\tfrac{1}{2}\frac{1}{\sigma}\frac{d}{d\sigma}\left(\sigma\,\overline{w_r^2w_z^2}\right)$$
$$+2U_\tau^4\sigma+\frac{1}{2\sigma}\frac{d}{d\sigma}\left(\sigma\,\overline{w_r^2}\right)\overline{w_z^3}, \qquad (3.6)$$

$$\overline{w_r^3}\,\frac{dU}{d\sigma} = -\beta_3 U_T^2\,\sigma\frac{dU}{d\sigma} - c_3 U_T^4 - \frac{1}{\sigma}\frac{d}{d\sigma}(\sigma\,\overline{w_r^3 w_z})$$

$$+ 2U_T^2\left[\frac{d}{d\sigma}(\sigma\,\overline{w_r^2}) + \overline{w_r^2} - \overline{w_\theta^2}\right], \qquad (3.7)$$

$$\overline{w_r w_\theta^2}\,\frac{dU}{d\sigma} = -\beta_3 U_T^2\,\sigma\frac{dU}{d\sigma} - c_3 U_T^4 - \frac{1}{\sigma^2}\frac{d}{d\sigma}(\sigma^2\,\overline{w_r w_z w_\theta^2})$$

$$+ 2U_T^2\,\overline{w_\theta^2}. \qquad (3.8)$$

Eliminating the triple correlation $\overline{w_z w_r^2}$ from (3.2) and (3.6) we get

$$\overline{w_r^2}\,\frac{dU}{d\sigma} + \frac{1}{\sigma}\frac{d}{d\sigma}\left\{\sigma\left(\frac{dU}{d\sigma}\right)^{-1}\left[-c_4 U_T^4\,\sigma - \frac{1}{2\sigma}\frac{d}{d\sigma}(\sigma\,\overline{w_r^2 w_z^2})\right.\right.$$

$$\left.\left. + 2U_T^4\,\sigma + \frac{1}{2\sigma}\frac{d}{d\sigma}(\sigma\,\overline{w_r^2})\overline{w_z^2}\right] - \beta_4 U_T^2\,\sigma\right\} = -a_4 U_T^2\frac{dU}{d\sigma}$$

$$-b_4 U_T^2\,\sigma + \frac{1}{\sigma}\overline{w_z w_\theta^2} - \frac{2\nu a}{\lambda^2}(m-2)U_T^2\,\sigma S. \qquad (3.9)$$

Applying the condition of determinateness[3] that the ratio of the mean velocity over the frictional velocity, U/U_T, does not depend upon the turbulent fluctuations and the Reynolds number of the flow, we find the terms that involve only the ratio U/U_T are connected by the following equation,

$$\frac{1}{\sigma}\frac{d}{d\sigma}\left\{\sigma\left[-\beta_4 - (c_4 - 2)\,U_T\,\sigma\Big/\frac{dU}{d\sigma}\right]\right\} = -\frac{a_4}{U_T}\frac{dU}{d\sigma} - b_4\,\sigma \qquad (3.10)$$

4. THE MEAN VELOCITY DISTRIBUTION.

Equation (3.10) determines the mean velocity distribution across the pipe. Let us introduce a dimensionless function χ defined by

$$\sigma U_T\Big/\frac{dU}{d\sigma} = A\chi. \qquad (4.1)$$

Then (3.10) becomes

$$\frac{\beta_4}{\sigma}+(c_4-2)A\left(\frac{\chi}{\sigma}+\frac{d\chi}{d\sigma}\right)=\frac{a_4\sigma}{A\chi}+b_4\sigma. \tag{4.2}$$

Multiplying (4.2) by $1/\sigma$ and writing By for σ^2, we get

$$\frac{\beta_4}{By}+\frac{(c_4-2)A}{B}\left(\frac{\chi}{y}+2\frac{d\chi}{dy}\right)=\frac{a_4}{A\chi}+b_4. \tag{4.3}$$

Since the constants A and B are arbitrary, we choose them such that

$$\frac{\beta_4}{B}=-\frac{(c_4-2)A}{B}=\frac{a_4}{A}. \tag{4.4}$$

Lat $A_4 b/a_4=-\lambda$, then (4.3) may be written as

$$\frac{1}{y}-\left(\frac{\chi}{y}+2\frac{d\chi}{dy}\right)=\frac{1}{\chi}-\lambda. \tag{4.5}$$

From the observed mean velocity distribution, it is seen that U is approximately parabolic in the central region of the pipe, so χ and $d\chi/dy$ must both be finite at $y=0$. Equation (4.5) shows that $\chi=1$ at $y=0$. With this boundary condition on χ, the solution of χ is completely determined except for the variable parameter λ which governs the general shape of the curve. In (4.5) let $y\to 0$ we find

$$\left(\frac{1-\chi}{y}\right)_{y=0}=-\left(\frac{d\chi}{dy}\right)_{y=0}=\frac{1}{3}(1-\lambda). \tag{4.6}$$

This relation will be useful in actual calculations.

The experimental results indicate that $\sigma d\sigma/dU$ is nearly a constant in the central part of the pipe. In the present theory we find

$$U_\tau\left[\frac{d}{d\sigma}\left(\sigma\frac{dU}{d\sigma}\right)\right]_{\sigma=0}=\left[\frac{d}{dy}(A\chi)\frac{dy}{d\sigma}\right]_{\sigma=0}$$

$$=\left[\frac{2A}{B}\sigma\frac{d\chi}{dy}\right]_{\sigma=0}=0.$$

The second derivative of $\sigma\, d\sigma/dU$ at the point $y=0$ is

$$U_1\left[\frac{d^2}{d\sigma^2}\left(\sigma\frac{dU}{d\sigma}\right)\right]_{\sigma=0} = \frac{2A}{B}\left(\frac{d\chi}{dy}\right)_{\sigma=0}.$$

To make the theory agree better with the experiment $\left[\frac{d^2}{d\sigma^2}\left(\sigma\frac{dU}{d\sigma}\right)\right]_{\sigma=0}$ should be nearly 0. If it is exactly zero, then $(d\chi/dy)_{y=0}=0$, and the only solution of (4.5) which is developable in a power series in y is $\chi=1$. We shall choose the value of λ such that $(d\chi/dy)_{y=0}$ is nearly equal to zero. The writer has computed χ for different values of λ near to 1, and has found that the final result is insensitive to the value of λ chosen. As a good approximation λ is set to be 0.9

With the value of λ determined, and the boundary conditions stated above equation (4.5) can be integrated numerically by Runge and Kutta's method. The result in tabulated in Table I below.

Table I

y	χ^2	y	χ^2	y	χ^2
0.0	1.000	2.0	0.793	4.0	0.279
0.5	0.984	2.5	0.716	4.5	0.113
1.0	0.919	3.0	0.610	4.4	0.043
1.5	0.865	3.5	0.471	4.46	0.000

Now we proceed to determine the constants A and B. The mean velocity distribution is given by

$$(U_m-U)/U_1 = -\frac{B}{2A}\int_0^y \frac{1}{\chi}\,dy. \tag{4.7}$$

Since near the wall U changes rapidly so that χ is nearly zero, we

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5 C. Runge and H. König, *Numerisches Rechnen* (1924), p. 295.

choose the value of B such that, at $y = 4.46$, $\sigma^2 = 1$, that is $B = 1/4.46$. From Table I the values of χ^2 is interpolated; then χ and $1/\chi$ are calculated and the integral on the right-hand side of (4.7) is computed by using Simpson's rule. The value of A is determined by letting the theoretical curve to pass through the experimental point at $\sigma = 0.7$, where $(U_c - U)/U_\tau$ is equal to 3.85. The result is plotted in Fig. 1 and compared with experiment.

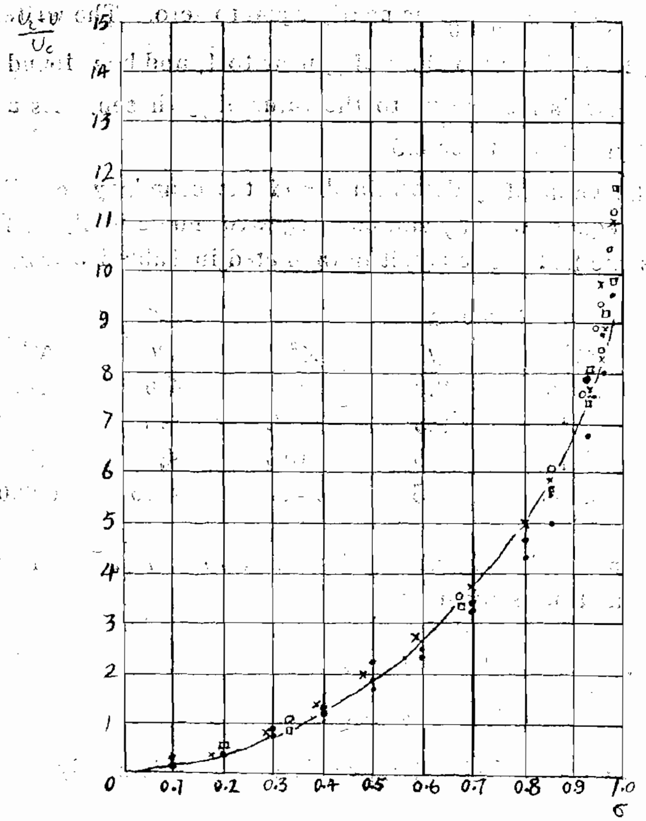

Fig. 1. Velocity distributions in a circular pipe

The experimental values for U are due to Stanton and Nikuradse and taken from a paper by Goldstein.[6] Results have also been obtained by Fage[7], but they agree with Stanton's measurements and are not shown to avoid confusion.

5. MEAN SQUARES OF VELOCITY FLUCTUATION

The integration of the equations (3.1), (3.3), (3.4), (3.5), (3.7), (3.8) can be carried out with the help of an approximate representation of the quadruple correlation given by Chou [3,4] together with the condition of determinateness[4] that the ratio of the mean squares of velocity fluctuations over U_τ^2 be independent both of the higher order correlation functions and of the Reynolds number of the mean flow. The result is

$$\frac{\overline{w_z^2}}{U_\tau^2}=\tfrac{3}{4}c_1+\tfrac{3}{2}(b_1-2\beta_1)\frac{\sigma^4}{U_\tau}\int_0^\sigma \sigma^{-4}\frac{dU}{d\sigma}\,d\sigma$$

$$+3(a_1+2)\frac{\sigma^4}{U_\tau^2}\int\sigma^{-6}\frac{dU}{d\sigma}\,d\sigma\int_0^\sigma \eta^2\frac{dU}{d\eta}\,d\eta, \qquad (5.1)$$

$$\frac{\overline{w_\theta^2}}{U_\tau^2}=\tfrac{3}{4}c_2+\tfrac{3}{4}(b_2-4\beta_2)\frac{\sigma^4}{U_\tau}\int_0^\sigma \sigma^{-4}\frac{dU}{d\sigma}\,d\sigma$$

$$+3a_2\frac{\sigma^4}{U_\tau^2}\int\sigma^{-8}\frac{dU}{d\sigma}\,d\sigma\int_0^\sigma \eta^4\frac{dU}{d\eta}\,d\eta, \qquad (5.2)$$

$$\frac{\overline{w_r^2}}{U_\tau^2}=\frac{c_2}{2}+\frac{4}{3\sigma^2}\int_0^\sigma \sigma\frac{\overline{w_\theta^2}}{U_\tau^2}\,d\sigma-\tfrac{1}{4}(b_2+2b_2-4\beta_2)\frac{1}{\sigma^2 U_\tau}\int_0^\sigma \sigma^3\frac{dU}{d\sigma}\,d\sigma$$

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6. S. Goldstein, *Proc. Roy. Soc.* A 159, 473 (1937).
7. A. Fage, *Phil. Mag.* (7) 21, 80 (1936).

$$-\frac{a_2+a_3}{U_\tau^2\sigma^2}\int_0^\sigma \frac{dU}{d\sigma}d\sigma\int_0^\sigma \eta^2\frac{dU}{d\eta}d\eta$$

$$+\frac{a_3}{\sigma^2 U_\tau^2}\int_0^\sigma \frac{1}{\sigma^2}\frac{dU}{d\sigma}d\sigma\int_0^\sigma \eta^4\frac{dU}{d\eta}d\eta. \qquad (5.3)$$

From the solution of $dU/d\sigma$ we see that for most part of the pipe the approximation $dU/d\sigma = \gamma U_\tau \sigma$ holds good. Introduction of this approximation into (5.1) and (5.2) yields

$$\frac{\overline{w_z^2}}{U_\tau^2} = \tfrac{3}{4}c_1 + \frac{3\gamma}{4}(2\beta_1 - b_1)\sigma^2 + \tfrac{3}{4}\gamma^2(a_1+2)\sigma^4(\ln\sigma + \epsilon_1), \qquad (5.4)$$

$$\frac{\overline{w_\theta^2}}{U_\tau^2} = \tfrac{3}{4}c_3 + \frac{3\gamma}{8}(4\beta_3 - b_3)\sigma^2 + \frac{a_3}{2}\gamma^2\sigma^4(\ln\sigma + \epsilon_3). \qquad (5.5)$$

Then (5.3) can also be evaluated, giving

$$\frac{\overline{w_r^2}}{U_\tau^2} = \tfrac{1}{2}(c_2 + c_3) + \frac{\gamma}{16}(8\beta_3 + 4\beta_2 - 3b_3 - 2b_2)\sigma^2$$

$$-\left[\frac{a_2}{24} + \left(\frac{7}{36} - \frac{\epsilon_2}{6}\right)a_3\right]\gamma^2\sigma^4 + \frac{a_3}{9}\gamma^2\sigma^4\ln\sigma. \qquad (5.6)$$

As pointed out by Chou[4] the values of the constants (invariable constants) $a_\mu, b_\mu, c_\mu, \beta_\mu$ ($\mu = 1,2,3,4$) (see § 3) should remain the same or at most can change very little for the different Reynolds number of the mean flow, but the constants of integration ϵ_1 and ϵ_2 can take arbitrary values depending upon the particular flow under consideration. We determine these constants by comparing them with the experimental measurements.

The existing experimental data gives values of $w_z'^2$, $w_r'^2$, $w_\theta'^2$ in terms U_0^2, where w_z', w_r', w_θ' are the maximum amplitudes of the fluctuations and U_0 is the mean velocity averaged over a cross-section. According to Fage

$$\frac{\overline{w_z'^2}}{\overline{w_z^2}} = \frac{\overline{w_r'^2}}{\overline{w_r^2}} = \frac{\overline{w_\theta'^2}}{\overline{w_\theta^2}}, \qquad U_\tau^2 / U_0^2 = 0.0042.$$

For the convenience of comparing with the experimental results we write $\left(\frac{w_z'}{U_0}\right)^2, \left(\frac{w_r'}{U_0}\right)^2, \left(\frac{w_\theta'}{U_0}\right)^2$ instead of $\frac{\overline{w_z^2}}{U_\tau^2}, \frac{\overline{w_r^2}}{U_\tau^2}, \frac{\overline{w_\theta^2}}{U_\tau^2}$ in equations (5.4), (5.5), (5.6). The constants are altered by multiplying a constant factor.

Since the experimental points are rather scattered, we obtain the values of the constants by drawing the theortical curve which should fit these points best instead of passing the curve through any particular points. The constants c_μ are determined directly by the values of the mean squares of fluctuation at the center of the pipe:

$$c_1 = \tfrac{4}{3}\left(\frac{w_z'}{U_0}\right)^2_{\sigma=0} \qquad c_3 = \tfrac{4}{3}\left(\frac{w_\theta'}{U_0}\right)^2_{\sigma=0},$$

$$c_2 = 2\left(\frac{w_r'}{U_0}\right)^2_{\sigma=0} - c_3 = \tfrac{2}{3}\left(\frac{w_r'}{U_0}\right)^2_{\sigma=0}.$$

The values of the various constants so determined are tabulated in table II below.

Table II

Constants	R = 6090	13440	18340
$\frac{3}{4}c_1$	.038	.050	.042
$3\gamma^2(2\beta_1+b_1)/4$	.067	.067	.067
$3\gamma^2(a_1+3)\epsilon_1/4$	.137	.100	.075
$3\gamma^2(a_1+2)/4$	.103	.108	.103

$-3c_3/4$	.009	.009	.009
$3\gamma(4\beta_3-b_3)/8$	.041	.041	.041
$a_3\gamma^2\epsilon_3/2$	.174	.174	.199
$a_3\gamma^2/2$	.170	.170	.100
$(c_2+c_3)/2$	.009	.009	.009
$\gamma(8\beta_3+A\beta_2-3b_3-2b_2)/16$	.015	.060	.055
$-\gamma^2[a_2/24+(\frac{7}{36}-\frac{\epsilon_2}{6})a_3]$	.067	.020	.020
$a_3\gamma^2/9$	.038	.022	.022

In the above table we have determined the invariable constants for the case $R=13440$ first for the mean square of velocity fluctuation $\overline{w_\theta^2}$ and $\overline{w_z^2}$ and then use these constants for the other two flows with Reynolds numbers 8090 and 18340. However this procedure does not work for $\overline{w_r^2}$; in this case all the invariable constants for the three Reynolds numbers have to be determined separately in order to fit the experiment.

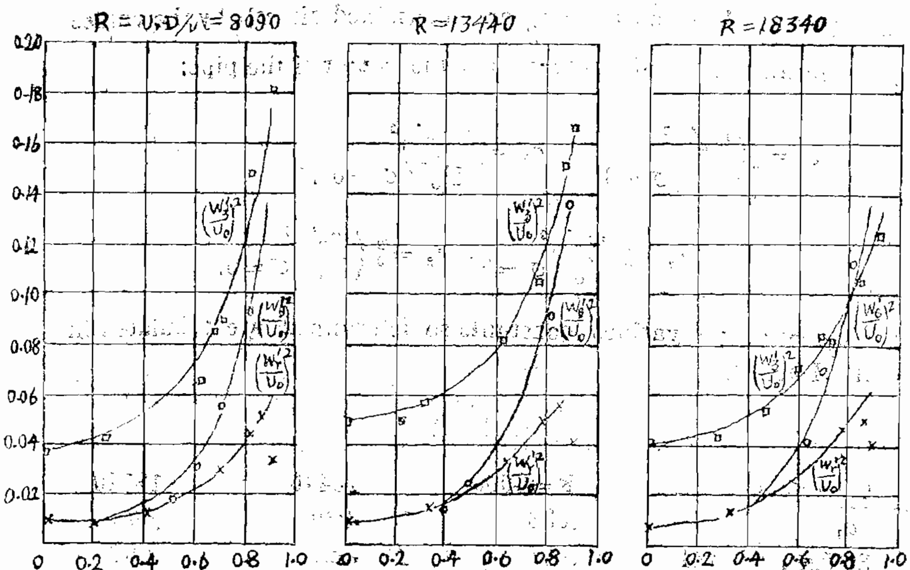

Fig. 2

Magnitudes of velocity fluctuation distributions in a circular pipe.

The theoretical values for the mean squares of the three fluctuation components are plotted in fig. 2 and compared with the experimental values.

The writer wishes to express his gratitude to Prof. P. Y. Chou for suggosting the problem and for his valuable advice in the course of investigation.