VELOCITY AND TEMPERATURE DISTRIBUTIONS IN TURBULENT WAKES BEHIND A ROW OF EQUALLY SPACED PARALLEL RODS AND BEHIND A SQUARE GRID

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ABSTRACT

Chou's theory of turbulence is applied successfully to the wakes behind a system of equal distant parallel rods and a square grid. It is found that both the velocity and temperature distributions in the first kind of wake are the same cosine functions of the coordinates across the wake and those for the second kind of wake are just the superpositions of the wakes behind two perpendicular systems of parallel rods. In both cases Taylor's law of linear decay of turbulent fluctuation and quadratic increase of scale of micro-turbulence are obtained.

1. *Introduction.*

In the preceding paper[1] Chou's theory of turbulence[2] has been applied successfully to the wakes behind an infinite cylinder and a body of revolution. In the course of solution we have introduced the assumption that the first derivatives of the triple correlations are constant across the wake. In order to test the validity of this assumption, let us consider the velocity and temperature distributions in the wake behind a row of equally spaced parallel rods in which the triple correlations play a very important part. It will be shown below that if the above assumption is used, the mean squared components of velocity fluctuations which are perpendicular to the

1. N. Hu, *Chinese J. Phys.* 5 (1944) 1—29.
2. P.Y. Chou. "On an Extension of Reynolds' Method of Finding Apparent Stresses and the Nature of Turbulence." *Chinese J. Phys.* 4 (1940) 1-33.

mean flow are constant across the wake. This important result can be verified by direct observation.

The velocity and temperature distributions are found to be the same cosine function of the coordinate across the wake. Their agreement with the experiment of Gran Olsson is found to be very good.

Since the parallel rods and grids are usually the main devices for producing isotropic turbulence discussed in Taylor's statistical theory, the turbulent flow behind a square grid is also investigated with a view to finding the nature of the turbulence thus produced and its relation with Taylor's theory. It is found that both the velocity and temperature distributions are just the superposition of the two flows behind the two systems of parallel rods perpencicular to each other. In both cases Taylor's laws of linear decay of turbulent fluctuation and quadratic increase of the scale of micro-turbulence are obtained.

2. *The wake behind a row of equally spaced parallel rods.*

We shall use the same notation as that of § 2 of the preceding paper and take the z-axis along the axis of any one of the rods and the x-axis in the direction of flow which is perpendicular to the plane of the rods. Let M be the distance between the neighbouring rods, then the x and y components of the mean velocity U and V behind the rods must be a periodic function of y, with period M, being minimum at $y=0, M, 2M, \ldots\ldots$, and maximum at $y=\frac{1}{2}M, \frac{3}{2}M, \frac{5}{2}M, \ldots\ldots$. Let U_0 be the average value of U over y, between $y=0$ and $y=\frac{1}{2}M$, the U_0-U must be zero at $y=\frac{1}{4}M, \frac{3}{4}M, \frac{5}{4}M, \ldots\ldots$. It is sufficient now to find U for $0\leq y\leq\frac{1}{4}M$ only, since U_0-U will be symmetrical about $y=0$ and antisymmetrical about $y=\frac{1}{4}M$. If we further assume that the mean distributions are similar in different cross-sections of the wake, then when x is large we may put

$$U=U_0-\frac{U_0}{x^q}F(y), \qquad V=\frac{U_0}{x^{q+1}}F_1(y). \tag{1}$$

where both q, a constant, and $F(y)$, the periodic function, are to be determined presently. That V must be of this form follows immediately from the equation of continuity,

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0.$$

Let the velocity fluctuation w^i be proportional to $1/x^s$ and $\overline{p_0-p}$ be proportional to $\overline{w_y{}^2}$, then the x component of Reynolds' equations of mean motion reads, after neglecting the small terms of higher powers of $1/x$, as

(2) $$qF=-f_2{}',$$

which gives immediately

(3) $$q+1=2s.$$

The constant q is usually determined from the integral of dragging force. In the present ease this force per unit length of each rod is given by (since $\int_{-\frac{M}{2}}^{+\frac{M}{2}}\frac{U_0{}^2}{x^q}F(y)dy=0$)

(4) $$G=\int_{-\frac{M}{2}}^{+\frac{M}{2}}\left\{\tfrac{1}{2}U_0{}^2-\tfrac{1}{2}U^2+(p_0-p)-\frac{1}{\rho}\tau_{xx}\right\}dy$$

$$=\int_{-\frac{M}{2}}^{+\frac{M}{2}}\left\{\frac{U_0{}^2}{2x^{2q}}[F(y)]^2+(\overline{p_0-p})+\frac{1}{\rho}\tau_{xx}\right\}dy,$$

where $\overline{p_0}$ is the mean pressure of the fluid before reaching the rods. Since $p_0-p\propto 1/x^{2s}$, then in order that G be independent of x, we must have

(5) $$2q=2s.$$

Comparing (5) with (3), we have

(9) $$s=1, \qquad q=1.$$

It will be noted that in deriving (6), we have used the condition that $\overline{p_0-p}\propto\overline{w_y{}^2}$. For the case of wakes discussed in the preceding paper this condition follows directly from the y component of Reynolds' equations of mean motion. Since in the present case, as we shall see later, all terms of this equation are zero separately (cf. the note of (13)), the validity of the above condition is questionable. But we can still obtain (6) by an alternative argument.

We may retain q as undetermined and put

$$
(7)\quad \begin{cases} -\dfrac{1}{\rho}\tau_{xx} = \dfrac{U_0^2}{x^{q+1}} f_1(y), & -\dfrac{1}{\rho}\tau_{xy} = \dfrac{U_0^2}{x^{q+1}} f_2(y), \\ -\dfrac{1}{\rho}\tau_{yy} = \dfrac{U_0^2}{x^{q+1}} f_3(y), & -\dfrac{1}{\rho}\tau_{zz} = \dfrac{U_0^2}{x^{q+1}} f_4(y), \end{cases}
$$

and

$$
(8)\quad \begin{cases} \overline{w_x^3} = \dfrac{U_0^3}{x^{\frac{3}{2}(q+1)}} h_0(y), & \overline{w_x^2 w_y} = \dfrac{U_0^3}{x^{\frac{3}{2}(q+1)}} h_1(y), \\ \overline{w_x w_y^2} = \dfrac{U_0^3}{x^{\frac{3}{2}(q+1)}} h_2(y), & \overline{w_y^3} = \dfrac{U_0^3}{x^{\frac{3}{2}(q+1)}} h_3(y), \\ \overline{w_x w_z^2} = \dfrac{U_0^3}{x^{\frac{3}{2}(q+1)}} h_4(y), & \overline{w_y w_z^2} = \dfrac{U_0^3}{x^{\frac{3}{2}(q+1)}} h_5(y). \end{cases}
$$

Substituting (1), (7) and (8) into the equations of double correlations (2,2) of the preceding paper, we have

$$
(9)\quad \begin{cases} \dfrac{2q}{x^{2q+2}} f_1 F - \dfrac{2}{x^{2q+2}} f_2 F' - \dfrac{q+1}{x^{q+2}} f_1 + \dfrac{q+1}{x^{2q+2}} F f_1 - \dfrac{\frac{3}{2}(q+1)}{x^{\frac{3}{2}(q+1)+1}} h_0 \\ \qquad + \dfrac{h_1'}{x^{\frac{3}{2}(q+1)}} = -\dfrac{2\nu k_{(11)}}{U_0\lambda^2} \dfrac{f_1}{x^{q+1}}, \\ -\dfrac{q}{x^{2q+2}} f_3 F' - \dfrac{q+1}{x^{q+2}} f_2 + \dfrac{q+1}{x^{2q+2}} F f_2 - \dfrac{\frac{3}{2}(q+1)}{x^{\frac{3}{2}(q+1)+1}} h_1 + \dfrac{h_2'}{x^{\frac{3}{2}(q+1)}} \\ \qquad = -\dfrac{2\nu k_{(12)}}{U_0\lambda^2} \dfrac{f_2}{x^{q+1}}, \\ -\dfrac{q+1}{x^{q+2}} f_3 + \dfrac{q+1}{x^{2q+2}} F f_3 - \dfrac{\frac{3}{2}(q+1)}{x^{\frac{3}{2}(q+1)+1}} h_2 + \dfrac{1}{x^{\frac{3}{2}(q+1)}} h_3' \\ \qquad = -\dfrac{2\nu k_{(22)}}{U_0\lambda^2} \dfrac{f_3}{x^{q+1}}, \\ -\dfrac{q+1}{x^{q+2}} f_4 + \dfrac{q+1}{x^{2q+2}} F f_4 - \dfrac{\frac{3}{2}(q+1)}{x^{\frac{3}{2}(q+1)+1}} h_4 + \dfrac{1}{x^{\frac{3}{2}(q+1)}} h_5' \\ \qquad = -\dfrac{2\nu k_{(33)}}{U_0\lambda^2} \dfrac{f_4}{x^{q+1}}. \end{cases}
$$

We note that in writing (9) terms involving F_1 have been neglected since they must be small quantities of higher order according to (1). From the second equation of (9) we see that the terms containing the lowest powers of $1/x$ are $-qf_5F'/x^{2q+1}$ and $(q+1)f_2/x^{q+2}$. From these two terms we can conclude that

$$(10) \qquad 2q+1 \leqq q+2 \quad \text{or} \quad q \leqq 1.$$

For if $q>1$, then we can neglect all terms involving higher powers of $1/x$ and the only terms left in that equation would be

$$-\frac{q+1}{x^{q+2}} f_2 = -\frac{2\nu k_{(12)}}{U_0\lambda^2} \frac{1}{x^{q+1}} f_2.$$

This, of course, will leave the function f_2 indeterminate. In other words q cannot be greater than unity necessarily.

The function

$$(11) \qquad l = \frac{2\nu x^q}{U_0\lambda^2}$$

must be a function of y only. Substituting (11) into the third and fourth equations of (9), we see that in order f_3 and f_4 to be different from zero, we must have $q=1$. Substituting this value of q into (3), we still obtain (6). Hence the assumption $\overline{p_0} - \overline{p} \propto \overline{w_y^2}$ is equivalent to the condition that the motion must be determinate by the equation of motion for the present problem.

Using the value of q and s given by (6) and putting

$$(12) \qquad \overline{p} = \frac{U_0^2}{x^2} f_0(y)$$

and neglecting the terms which are of higher power of $1/x$, we can reduce the Reynolds' equations of mean motion and the equations of double correlations respectively into the following form

$$(13) \qquad F = -f_2, \quad 0 = f_0' + f_3',^* \quad \left(-\frac{1}{\rho}\overline{p} = \frac{U_0^2}{x^2} f_0(\eta)\right)$$

and

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*It will be shown later that f_0' and f_3' vanish separately as f_3 is found to be constant. It is for this reason that the relation $\overline{p_0} - \overline{p} \propto \overline{w_y^2}$ cannot be proved rigorously.

$$(14)\quad \begin{cases} 2f_2 F' - h_1' = (k_{(11)} l - 2) f_1, \\ f_3 F' - h_2' = (k_{(12)} l - 2) f_2, \quad - h_3' = (k_{(22)} l - 2) f_3, \\ - h_5' = (k_{(33)} l - 2) f_4, \quad l = 2\bar{\nu}\bar{x}/U_0 \lambda^2. \end{cases}$$

Now we may introduce the assumption which we have used before that the functions h_i' and l are constants across the wake. Sence $F'(0) = f_2(0) = 0$ and $f_1(0)$, $f_3(0)$ and $f_4(0)$ and all different from zero, then from (14) we have

$$(15)\qquad h_2' = 0, \qquad h_1', \ h_3' \ h_5' \neq 0.$$

Therefore (14) become

$$(16)\quad \begin{cases} f_1 = -\dfrac{h_1'}{k_{(11)} l - 2} + \dfrac{2f_2 F'}{k_{(11)} l - 2}, \quad f_2 = \dfrac{f_3 F'}{k_{(12)} l - 2} = \dfrac{1}{k^2} F', \\ f_3 = -\dfrac{h_3'}{k_{(22)} l - 2} = \text{const.}, \quad f_4 = -\dfrac{h_5'}{k_{(33)} l - 2} = \text{const.}, \end{cases}$$

where $k^2 = (k_{(12)} l - 2)/f_3$ is a positive constant.

Substituting the second equation of (16) into the first equation of (13), we have

$$(17)\qquad F = -\frac{d}{dy}\left(\frac{F'}{k^2}\right) = -\frac{1}{k^2} F''.$$

which, on integration, gives

$$(18)\qquad F(y) = A \cos ky + B \sin ky.$$

Now as $F = 0$ at $y = \frac{1}{4}M$ and $F' = 0$ at $y = 0$, then $B = 0$, $k^{-1} = M/2\pi$ and (18) becomes*

$$(19)\qquad F(y)/F(0) = \cos(2\pi y/M).$$

With similar process, the temperature distribution can be found. Thus let

$$(20)\qquad \bar{\theta}_0 - \bar{\theta} = \frac{U_0^2}{x}\phi(y)$$

and

$$l_1 = \frac{2\nu_1 x}{U_0 \lambda^2}, \tag{21}$$

where $\bar{\theta}$ is the mean temperature, $\bar{\theta}_0$, is the value of $\bar{\theta}$ at $y=0$, $\phi(y)$ is the unknown function to be determined from the theory and ν_1 has the similar meaning as in the preceding paper, then using (3.7) and (3.8) of that paper and putting

$$\overline{w_x\theta'} = \frac{U_0^2}{x^2} f_5(y), \quad \overline{w_y\theta'} = \frac{U_0^2}{x^2} f_6(y), \quad \overline{w_z\theta'} = \frac{U_0^2}{x^2} f_7(y),$$

$$\overline{w_x^2\theta'} = \frac{U_0^3}{x^3} h_6(y), \quad \overline{w_y w_x\theta'} = \frac{U_0^3}{x^3} h_7(y), \quad \overline{w_y^2\theta'} = \frac{U_0^3}{x^3} h_8(y),$$

we have, on neglecting terms of higher powers of $1/x$,

$$\phi = -f_6', \qquad f_2 \phi' = (k_{(2)} l_1 - 2) f_6,$$

corresponding to the first equation of (13) and the second equation of (14) respectively. The above equations can be solved in the same way as before and we obtain the resulting expression for temperature distribution in the following form also:

$$\phi(y)/\phi(0) = \cos\frac{2\pi}{M} y, \tag{22}$$

where $M = 2\pi\,(k_{(2)}\, l_1 - 2)/f_2$.

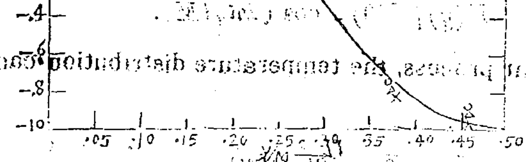

Fig. 1.—Velocity distribution. o: x = 75 cm, Δ: x = 100 cm, ×: x = 125 cm.

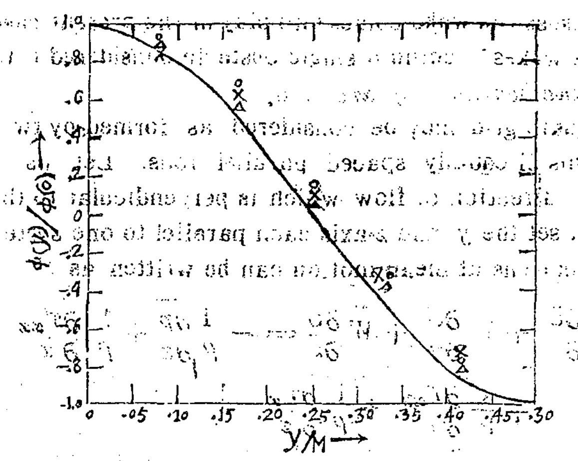

Fig. 2.—Temperature distribution. Δ: x = 7.5 cm. $\times$: x = 15 cm, o: x = 25 cm.

Figures 1 and 2 are the experimental velocity and temperature distributions obtained by Gran Olsson[3], who using the assumption based upon Prandtl's suggestion which is actually equivalent to the second equation of (16), obtains the same results (19) and (22). The comparison of (19) and (22) with observation has already been made in his original paper. It is seen from the figures that the agreement of both the shape of the distribution curves and the value of q with experiment are very good.

Two interesting points of the present solution will be noted here. In the first place, our assumption that the functions h_i' are constants leads directly to the result that the mean squares of the y and z components of turbulent velocity fluctuation should be constant across the wake. This can be verified by direct observation and taken as a crucial test of our original assumption. Secondly, substituting (1) and (5) into the second equation of (16), we obtain

$$\tau_{xy} = -\frac{\tau_{yy}\, x}{(k_{(12)}\, l - 2)U_0}\frac{\partial U}{\partial y} = (\text{const.})\frac{\partial U}{\partial y}. \tag{23}$$

We see that here again a constant coefficient of apparent

3. R. Gran Olsson, *Zeit. f. Angew. Math. u. Mech.* 16 (1936), 257-274.

viscosity across the wake comes into play in the present case as in the jets and the wakes[4] behind a single obstacle considered before.

3. *The wake behind a square grid.*

The square grid may be considered as formed by two perpendicular systems of equally spaced parallel rods. Let us choose the x-axis in the direction of flow which is perpendicular to the plane of the grid and set the y- and z-axis each parallel to one system of rods, then the equations of mean motion can be written as

$$
(24)\quad \begin{cases}
U\dfrac{\partial U}{\partial x}+V\dfrac{\partial U}{\partial y}+W\dfrac{\partial U}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial \bar{p}}{\partial x}+\dfrac{1}{\rho}\dfrac{\partial \tau_{xx}}{\partial x}\\
\qquad +\dfrac{1}{\rho}\dfrac{\partial \tau_{xy}}{\partial y}+\dfrac{1}{\rho}\dfrac{\partial \tau_{xz}}{\partial z}\\
U\dfrac{\partial V}{\partial x}+V\dfrac{\partial V}{\partial y}+W\dfrac{\partial V}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial \bar{p}}{\partial y}+\dfrac{1}{\rho}\dfrac{\partial \tau_{xy}}{\partial x}+\dfrac{1}{\rho}\dfrac{\partial \tau_{yy}}{\partial y}\\
U\dfrac{\partial W}{\partial x}+V\dfrac{\partial W}{\partial y}+W\dfrac{\partial W}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial \bar{p}}{\partial z}+\dfrac{1}{\rho}\dfrac{\partial \tau_{xz}}{\partial x}+\dfrac{1}{\rho}\dfrac{\partial \tau_{zz}}{\partial z}
\end{cases}
$$

and the equations of double correlations become

$$
(25)\quad \begin{cases}
-\dfrac{2}{\rho}\left(\dfrac{\partial U}{\partial x}\tau_{xx}+\dfrac{\partial U}{\partial y}\tau_{xy}+\dfrac{\partial U}{\partial z}\tau_{xz}\right)-\dfrac{1}{\rho}U\dfrac{\partial \tau_{xx}}{\partial x}\\
\qquad -\dfrac{1}{\rho}V\dfrac{\partial \tau_{xx}}{\partial y}-\dfrac{1}{\rho}W\dfrac{\partial \tau_{xx}}{\partial z}+\dfrac{\partial}{\partial x}\overline{w_x^3}\\
\qquad +\dfrac{\partial}{\partial y}\overline{w_x^2 w_y}+\dfrac{\partial}{\partial z}\overline{w_x^2 w_z}=\dfrac{2\nu k_{(11)}}{\rho\lambda^2}\tau_{xx}\\
-\dfrac{1}{\rho}\left(\dfrac{\partial U}{\partial x}\tau_{xy}+\dfrac{\partial V}{\partial x}\tau_{xx}+\dfrac{\partial U}{\partial y}\tau_{yy}+\dfrac{\partial V}{\partial y}\tau_{xy}+\dfrac{\partial V}{\partial z}\tau_{xz}\right)\\
\qquad -\dfrac{1}{\rho}U\dfrac{\partial \tau_{xy}}{\partial x}-\dfrac{1}{\rho}V\dfrac{\partial \tau_{xy}}{\partial y}-\dfrac{1}{\rho}W\dfrac{\partial \tau_{xy}}{\partial z}\\
\qquad +\dfrac{\partial}{\partial x}\overline{w_x^2 w_y}+\dfrac{\partial}{\partial y}\overline{w_x w_y^2}+\dfrac{\partial}{\partial z}\overline{w_x w_y w_z}\\
\qquad =\dfrac{2\nu k_{(12)}}{\rho\lambda^2}\tau_{xy}
\end{cases}
$$

4. C. C. Lin, "Velocity and Temperature Distributions in Turbulent Jets," *Sci. Rep. National Tsing Hua Univ.* (1941)

$$
\left\{
\begin{aligned}
& -\frac{2}{\rho^2}\left(\frac{\partial V}{\partial x}\tau_{xy}+\frac{\partial V}{\partial y}\tau_{yy}\right)-\frac{1}{\rho}U\frac{\partial \tau_{yy}}{\partial x}-\frac{1}{\rho}V\frac{\partial \tau_{yy}}{\partial y} \\
& \qquad -\frac{1}{\rho}W\frac{\partial \tau_{yy}}{\partial z}+\frac{\partial}{\partial x}\overline{w_x w_y^2}+\frac{\partial}{\partial y}\overline{w_y^3} \\
& \qquad +\frac{\partial}{\partial z}\overline{w_z w_y^2}=\frac{2\nu k(2)}{\rho\lambda^2}\tau_{yy}
\end{aligned}
\right.
$$

and two more equations obtained by interchanging the second and third components of coordinates and velocities in the above equations. In the above equations U, V and W are the x, y and z components of mean velocity, respectively. In (24) and (25) we have omitted the terms involving τ_{yz}, which must evidently vanish due to symmetry.

If we choose the intersection of the axis of any two perpendicular rods as the origin, then U will be a periodic function of both y and z with the same period equal to the mesh distance M of the grid (which is defined as the distance between the neighboring parallel rods). Therefore we have

$$\frac{\partial U}{\partial y}=0 \quad \frac{\partial^2 U}{\partial y^2}\neq 0 \text{ along } y=\tfrac{1}{2}nM \; (U \text{ is max. when } n \text{ is odd, min. when } n \text{ is even})$$

$$\frac{\partial U}{\partial z}=0 \quad \frac{\partial^2 U}{\partial z^2}\neq 0 \text{ along } z=\tfrac{1}{2}mM \; (U \text{ is max. when } m \text{ is odd, min. when } m \text{ is even})$$

$(n, m=0, 1, 2, 3, \ldots)$

Let U_0 be defined by

$$U_0=\frac{4}{M^2}\int_0^{\frac{M}{2}}\int_0^{\frac{M}{2}} U\,dy\,dz.$$

It is sufficient to consider the region $0 \leq y \leq \frac{1}{2}M$, $0 \leq z \leq \frac{1}{2}M$ only, since U_0-U must be symmetrical about $y=0$ and $z=0$ and also about $y=\frac{1}{2}M$ and $z=\frac{1}{2}M$. We put as before

$$
(26)\quad
\left\{
\begin{aligned}
& U_0-U=\frac{U_0}{x^q}F(y,z), \quad V=\frac{U_0}{x^{q+1}}F_2(y,z), \\
& W=\frac{U_0}{x^{q+1}}F_3(y,z).
\end{aligned}
\right.
$$

$$(27)\quad \begin{cases} -\dfrac{1}{\rho}\tau_{xx} = \dfrac{U_0^2}{x^{2s}} f_1(y,z), & -\dfrac{1}{\rho}\tau_{xy} = \dfrac{U_0^2}{x^{2s}} f_2(y,z), \\ -\dfrac{1}{\rho}\tau_{yy} = \dfrac{U_0^2}{x^{2s}} f_3(y,z), & \\ -\dfrac{1}{\rho}\tau_{zz} = \dfrac{U_0^2}{x^{2s}} f_4(y,z), & -\dfrac{1}{\rho}\tau_{xz} = \dfrac{U_0^2}{x^{2s}} f_5(y,z), \\ -\dfrac{1}{\rho}\overline{\tau_{yz}} = \dfrac{U_0^2}{x^{2s}} f_6(y,z) & \end{cases}$$

and

$$(28)\quad \begin{cases} \overline{w_x^3} = \dfrac{U_0^3}{x^{3s}} h_0(y,z), & \overline{w_x^2 w_y} = \dfrac{U_0^3}{x^{3s}} h_1(y,z) \\ \overline{w_y w_x^2} = \dfrac{U_0^3}{x^{3s}} h_2(y,z), & \overline{w_y^3} = \dfrac{U_0^3}{x^{3s}} h_3(y,z), \\ \overline{w_x w_z^2} = \dfrac{U_0^3}{x^{3s}} h_4(y,z), & \overline{w_y w_z^2} = \dfrac{U_0^3}{x^{3s}} h_5(y,z), \\ \overline{w_x w_y w_z} = \dfrac{U_0^3}{x^{3s}} h_6(y,z), & \\ \overline{w_x^2 w_z} = \dfrac{U_0^3}{x^{3s}} h_7(y,z), & \overline{w_z w_y^2} = \dfrac{U_0^3}{x^{3s}} h_8(y,z) \\ \overline{w_z^3} = \dfrac{U_0^3}{x^{3s}} h_9(y,z); & \end{cases}$$

and from the equation of motion we have $q+1=2s$. If we further assume that $\overline{p_0}-\overline{p} \propto \overline{w_y^2}$, then since the flow of momentum

$$(29)\quad \int_{-\frac{M}{2}}^{+\frac{M}{2}}\int_{-\frac{M}{2}}^{+\frac{M}{2}} \left\{ \frac{1}{2}U_0^2 - \frac{1}{2}U^2 + \frac{1}{\rho}(\overline{p_0}-\overline{p}) - \frac{1}{\rho}\tau_{xx} \right\} dydz$$

$$= \int_{-\frac{M}{2}}^{\frac{M}{2}}\int_{-\frac{M}{2}}^{\frac{M}{2}} \left\{ \frac{1}{2}U_0^2 \frac{1}{x^{2q}} F_1[(y,z)] + \frac{1}{\rho}(\overline{p_0}-\overline{p}) - \frac{\tau_{xx}}{\rho} \right\} dy\, dz$$

must be constant in every cross-section, we have as before

(30) $\qquad q = 1, \qquad s = 1.$

The alternative derivation based upon the equations of double correlation analogous to (9) also leads to the same result.

Using the above values for q and s, sucsutituting (26), (27) and (28) into (24) and (25) and putting

$$-\frac{\overline{p}}{\rho} = \frac{U_0^{\,2}}{x^2} f_0(y, z)$$

and retaining only the terms which involve the lowest powers of $1/x$, we have

$$(31) \qquad F_1 = -\frac{\partial f_2}{\partial y} - \frac{\partial f_5}{\partial z}, \qquad 0 = -\frac{\partial f_0}{\partial y} - \frac{\partial f_3}{\partial y},$$
$$0 = -\frac{\partial f_0}{\partial z} - \frac{\partial f_4}{\partial z}.$$

and

$$(32) \quad \begin{cases} 2\dfrac{\partial F_1}{\partial y} f_2 + 2\dfrac{\partial F_1}{\partial z} f_5 - \dfrac{\partial h_1}{\partial y} - \dfrac{\partial h_7}{\partial z} = (k_{(11)}\, l - 2) f_1, \\ \dfrac{\partial F_1}{\partial y} f_3 - \dfrac{\partial h_2}{\partial y} - \dfrac{\partial h_6}{\partial z} = (k_{(12)}\, l - 2) f_2, \\ -\dfrac{\partial h_3}{\partial y} - \dfrac{\partial h_8}{\partial z} = (k_{(22)}\, l - 2) f_3, \\ -\dfrac{\partial h_5}{\partial y} - \dfrac{\partial h_9}{\partial z} = (k_{(33)}\, l - 2) f_4, \\ \dfrac{\partial F_1}{\partial z} f_4 - \dfrac{\partial h_6}{\partial y} - \dfrac{\partial h_4}{\partial z} = (k_{(13)}\, l - 2) f_5, \end{cases}$$

where $k = 2\nu x/\lambda^2$.

Now we shall introduce the same assumptions which we have used before into equations (32), namely, that l and the first derivatives of the function h_i are all constants. Since $F_1'(0) = f_2(0) = f_5(0) = 0$ and $f_1(0)$, $f_3(0)$ and $f_4(0)$ are all different from zero, the we have

$$(33)\quad \begin{cases} \dfrac{\partial h_2}{\partial y}+\dfrac{\partial h_6}{\partial z}=\dfrac{\partial h_3}{\partial y}+\dfrac{\partial h_4}{\partial z}=0, & \dfrac{\partial h_3}{\partial y}+\dfrac{\partial h_8}{\partial z}=-b_1=\text{const}, \\ \dfrac{\partial h_1}{\partial y}+\dfrac{\partial h_7}{\partial z}=-a=\text{const}, & \dfrac{\partial h_5}{\partial y}+\dfrac{\partial h_9}{\partial z}=-b_2=\text{const}. \end{cases}$$

Since all functions must be symmetrical with respect to y and z, we have $b_1=b_2$. By the condition that $k(22)=k(33)$ and $k(12)=k(13)$, (32) can further be reduced to the following forms :

$$(34)\quad \begin{cases} f_1=\dfrac{a}{k(11)l-2}+\dfrac{2f_2}{k(11)l-2}\left(\dfrac{\partial F_1}{\partial y}+\dfrac{\partial F_1}{\partial z}\right), \\ f_2=\dfrac{f_3}{k(12)l-2}\dfrac{\partial F_1}{\partial y}=\dfrac{1}{k^2}\dfrac{\partial F_1}{\partial y}, \\ f_5=\dfrac{f_4}{k(13)l-2}\dfrac{\partial F_1}{\partial z}=\dfrac{1}{k^2}\dfrac{\partial F_1}{\partial z}, \quad \left(k^2=\text{positive constant}\right), \\ f_3=f_4=\dfrac{b_1}{k(22)l-2}=\text{const.}. \end{cases}$$

Eliminating f_2 and f_5 between (34) and the first equation of (31), we obtain

$$(35)\qquad \frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}+k^2F_1=0.$$

A particular solution of (35) is

$$(36)\qquad F_1=F_\mu=\pm\cos(\mu y-\xi_\mu)\cos(\mu' z-\zeta_\mu),$$

where

$$(37)\qquad \mu^2+\mu'^2=k^2,$$

and ξ_μ, ζ_μ are constants. The condition of symmetry $F(y,z)=F(y,-z)=F(-y,z)$, gives immediately $\xi_\mu=\zeta_\mu=0$. Then (36) becomes

$$(38)\qquad F_\mu=\pm\cos\mu y\cos\mu' z.$$

The boundary conditions may be stated as that F_1 is maximum at $y=mM$, $z=nM$ and minimum at $y=(m+\frac{1}{2})M$, $z=(n+\frac{1}{2})M$ where n and m are integers or zero. Now since maximum and minimum values of F_{μ} are $+1$ and -1 respectively, the above conditions may be expressed more conveniently as follows:

(39) $F=\pm 1$ for $y=mM$, $z=nM$,

(40) $F=-1$ for $y=(m+\frac{1}{2})M$, $z=(n+\frac{1}{2})M$.

Substituting (39) into (38), we have

(41) $$\pm 1 = \cos \mu' mM \cos \mu nM.$$

It followe immediately what $\cos \mu' mM$ and $\cos \mu nM$ must be equal to unity separately and hence that

(42) $$\mu=\pm \frac{s'\pi}{M}, \quad \mu'=\pm \frac{t'\pi}{M},$$

where s' and t' are positive integers or zero. We also see from (41) that the negative sign in the left-hand side is impossible since this would require that one of $s'm$ and $t'n$ must be even and the other must be odd while m and n are any integers. On the other hand, the positive sign is permissible for we can choose s' and t' to be both even such that $\cos \mu mM$ and $\cos \mu' nM$ always have the same sign.

Next we shall introduce the second condition (40) into (41). Then we have

$$-1=\cos \mu'(m+\tfrac{1}{2})M \cos \mu(n+\tfrac{1}{2})M.$$

It follows that

(43) $$\cos \mu(n+\tfrac{1}{2})M=-\cos \mu'(m+\tfrac{1}{2})M=\pm 1.$$

Then we have

(44) $$\mu=\pm \frac{2s''\pi}{M}, \quad \mu'=\pm \frac{2t''\pi}{M},$$

where s'' and t'' are positive integers or zero. Comparing (43) with

(44), we obtain

$$(45)\qquad s'=2s'', \qquad t'=2t''.$$

Hence s' and t' are all even, a result which is consistent with our foregoing requirement.

We also see from (43) that $\mu(m+\frac{1}{2})$ and $\mu'(n+\frac{1}{2})$ cannot be both even or both odd. From (44) it follows that s'' and t'' cannot be both even or both odd. Thus we may differentiate the following two cases

(i) $\qquad s''=2s, \qquad t''=2t+1;$

(ii) $\qquad s''=2t+1, \quad t''=2s,$

where s and t are positive integers or zero.

We shall first consider the case (i). Using (44) we may write (38) and (37) respectively into the following forms;

$$(46)\qquad F_{\mu}^{(i)}=\cos\frac{4s\pi}{M}y\cos\frac{(4t+2)\pi}{M}z,$$

$$(47)\qquad 4^2s^2+(4t+2)^2=k_{\mu}^2M^2/\pi^2.$$

For case (ii) it is only necessary to interchange the quantities y and z. Thus we have

$$(48)\qquad F_{\mu}^{(ii)}=\cos\frac{(4t+2)\pi}{M}y\cos\frac{4s\pi}{M}z,$$

where s and t are still connected by the relation (47).

A more general form of F_{μ} is the linear combination of (46) and (48):

$$(49)\qquad F_{\mu}=A_{\mu}\cos\frac{4s\pi}{M}y\cos\frac{(4t+2)\pi}{M}z$$
$$+B_{\mu}\cos\frac{(4t+2)\pi}{M}y\cos\frac{4s\pi}{M}z.$$

Since any solution of (35) must be symmetrical with respect to y and z in the present problem, we have $A_{\mu}=B_{\mu}$. Besides, there may exist more than one pairs of integers (s, t) satisfying (47). Thus the general form of F_1 may be written as

$$(50) \quad F_1 = \sum_{(s,t)} A_{(s,t)} \left\{ \cos \frac{4s\pi}{M} y \cos \frac{(4t+2)\pi}{M} z \mp \cos \frac{(4t+2)\pi}{M} y \cos \frac{4s\pi}{M} z \right\},$$

where the summation is taken over all pairs of integers (s, t) satisfying (47).

It will be noticed that k^2M^2 must be independent of both M and the Reynolds number according to (47) as the distribution of mean velocity must remain similar when both the value of M and the Reynolds number are changed. If the value of k^2M^2 is known then the form of (50) can be determined uniquely from (47). Now k^2M^2 is also given by the following relation

$$(51) \quad k^2M^2 = \frac{k_{(12)} l - 2}{f_{12}} M^2.$$

Using the third equation of (7) and the last equation of (14), we have

$$(52) \quad k^2M^2 = \left\{ k_{(12)} \frac{2\nu x}{\lambda^2 U_0} \right\} \Big/ \left\{ \frac{\overline{w_y^2}}{U_0^2} \left(\frac{x}{M} \right)^2 \right\}.$$

It is seen that the quantities in the right-hand side of above equation are all measurable quantities except the constant $k_{(12)}$, which may be either eliminated by two sets of observations if $2\nu x/\lambda^2 U_0$ and $(\overline{w_y^2}/U_0)(x/M)^2$ are not constant separately or equated to the same value obtained in the case of flow behind a system of parallel rods. Therefore the order of magnitude of k^2M^2 can be determined and compared with the value obtained by the following augument.

Comparing the result of last section, we find that the value of k^2M^2 for the wake behind the system of parallel rods is also given by the same equation (52). Since the quantities in the right-hand side of (52) must have the same order of magnitude for both cases, it follows that the values of k^2M^2 for the two cases must also have the same order. Now from the result $k^2M^2/\pi^2=4$ of the last section we see that the same quantity for the flow behind the grid cannot have a value greater than 100. The following table gives all the possible forms of F for $k\ M^2/\pi^2 \leqq 100$. It is seen that the form given in the second, third and fifth rows of that table do not take M as period, so they must be excluded. Now we introduce the further condition that F_1 cannot vanish more than once along the curve $y=z$ within the interval $0<y<\frac{1}{2}M$. It can easily be seen that the forms given in the fourth

Table 1

k^2M^2/π^2	all possible solutions (s,t) of (47)	F_1
4	(0,0)	$A\cos\frac{2\pi y}{M}+A\cos\frac{2\pi z}{M}$
16	(1,0)	$A\cos\frac{4\pi y}{M}\cos\frac{2\pi z}{M}+A\cos\frac{4\pi z}{M}\cos\frac{2\pi y}{M}$
36	(0,1)	$A\cos\frac{6\pi y}{M}+A\cos\frac{6\pi z}{M}$
52	(1,1)	$A\cos\frac{4\pi y}{M}\cos\frac{6\pi z}{M}+A\cos\frac{6\pi y}{M}\cos\frac{4\pi z}{M}$
68	(2,0)	$A\cos\frac{2\pi z}{M}\cos\frac{8\pi y}{M}+A\cos\frac{8\pi z}{M}\cos\frac{2\pi y}{M}$
100	(0,2),(2,1)	$A\cos\frac{10\pi y}{M}+A\cos\frac{10\pi z}{M}+B\cos\frac{8\pi y}{M}\cos\frac{6\pi z}{M}+B\cos\frac{6\pi y}{M}\cos\frac{8\pi z}{M}$

and sixth rows must also be excluded. Therefore the only solution which satisfies the above the conditions is

$$F_1=A\cos\frac{2\pi y}{M}+A\cos\frac{2\pi z}{M}. \tag{53}$$

Equation (53) can also be written in the following form

$$F_1(y,z)/F_1(0,0) = \tfrac{1}{2}\cos\frac{2\pi}{M}y + \tfrac{1}{2}\cos\frac{2\pi}{M}z. \tag{54}$$

We see that the distribution of velocity is just the superposition of the wakes behind two rows of parallel rods perpendicular to each other.

The temperature distribution is also found by the same process to be of the following form

$$\phi(y,z)/\phi(0,0) = \tfrac{1}{2}\cos\frac{2\pi y}{M} + \tfrac{1}{2}\cos\frac{2\pi z}{M}, \tag{55}$$

where $\phi(y,z)$ has the similar meaning as $\phi(y)$ defined by (20).

It is seen from the foregoing treatment that in both kinds of wakes the root-mean-square values of turbulent velocity fluctuations decrease as $1/x$. This is the linear law formulated by Taylor for the flow behind the grid[5]. If we put

$$\frac{\overline{w_y^{\,2}}}{U_0^{\,2}}\left(\frac{x}{M}\right)^2 = a, \tag{56}$$

then from (52) we have

$$\frac{2\nu x}{\lambda^2 U_0} = \frac{k^2M^2a+2}{k_{(12)}} = b, \text{ say.} \tag{57}$$

Since k^2M^2 is independent of U_0 and M, it follows that if a is independent of U_0 and M, b must also be so. Eliminating x between (56) and (57), we have

$$\frac{\lambda}{M} = \left(\frac{2a}{b}\right)^{\frac{1}{2}} \sqrt{\frac{\nu}{M\sqrt{\overline{w_y^{\,2}}}}} = A\sqrt{\frac{\nu}{M\sqrt{\overline{w_y^{\,2}}}}}, \tag{57}$$

where A will be independent of U_0 and M (i.e. a universal constant) if a is so. The above result is just the same as obtained by Taylor in his statistical theory of isotropic turbulence.

Our foregoing results shows that the turbulence produced behind

5. G. I. Taylor, *Proc. Roy. Soc. A*, *151* (1935), 421-478.

a square grid is not isotropic although it obeys all the laws deduced from the assumption of isotropy. The validity of the above two predictions (54) and (55) should be verifiable by direct observation. The accompanying result, $k_0^2 M^2/\Pi^2 = 4$, may also be compared with that calculated from (52) using experimental data in order to test the consistency of the present theory.

In conclusion, I wish to express my thanks to Professor P. Y. Chou for suggesting to me the present problem and for his kind help.