ON THE FACTORIZATION METHOD FOR QUANTUM MECHANICAL EIGENVALUE PROBLEMS

H. C. Lee (李華宗)

National Wuhan University, Loshan

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ABSTRACT

The Schrodinger-Infeld factorization method[1] for solving eigenvalue problems in quantum mechanics is very simple for the determination of the eigenvalues and eigenfunctions, without using power series or polynomial developments. However, no general theory underlying this method has been developed. It is the purpose of the present paper to formulate explicit conditions (A, B, C) under which the method applies, and to lay down formulas which will enable one to write out at once the solution of any eigenvalue problem satisfying these conditions. As illustrations the spherical harmonics, the Kepler problem in ordinary space and in spherical space, and the harmonic oscillator, are re-treated here on the basis of the theory now developed.

1. Introduction

Our question concerns Sturm-Liouville's eigenvalue problem on a differential equation of the type

$$\frac{d}{du}\left(P\frac{dv}{du}\right)+(\lambda G+Q)\,v=O$$

where λ is a parameter and the ratio G/P is positive and exists everywhere in the u-interval under consideration. By the changes of dependent and independent variables

$$y=(GP)^{1/4}\,v,\qquad x=\int\left(\frac{G}{P}\right)^{1/2}du,$$

Sturm-Liouville's equation is reduced to the canonical form

$$\frac{d^2y}{dx^2}+(\lambda+f)y=O \tag{1}$$

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1. L. Infeld, Phys. Rev. 50, 737 (1941).

We shall assume, as is the case in most eigenvalue problems in quantum mechanics, that f in (1) involves besides the variable x a constant m which can take the integral values

$$m = m_0, \; m_0+1, \; m_0+2, \ldots, \; +\infty,$$

where m_0 is a finite integer or eventually $-\infty$. Thus f may be written in the form $f(x, m)$.

By an eigensolution of (1) we mean any regular solution y of it, not identically zero, satisfying the boundary condition $y = 0$ at the limits of the x-interval. If the x-interval is infinite we impose the additional condition that the integral $\int y^2 \, dx$ extended over the whole x-interval is finite.

For each integral value of m, eigensolutions of (1) exist in general not for all but only for some particular values of λ which we call the *eigenvalues* of (1).

2. The Factorization Hypothesis

Suppose that (1) can be factorized in two ways as follows:

$$\left[\frac{d}{dx} - g(x, m)\right]\left[\frac{d}{dx} + g(x, m)\right] y + [\lambda - k(m)]\, y = 0$$

$$\left[\frac{d}{dx} + g(x, m+1)\right]\left[\frac{d}{dx} + g(x, m+1)\right] y + [\lambda - k(m+1)]\, y = 0$$

where $g(x, m)$ is a function of x involving m and $k(m)$ a constant depending upon m. If for brevity we introduce the differential operators

$$F_m = g(x, m) + \frac{d}{dx}, \quad G_m = g(x, m) - \frac{d}{dx}$$

the above factorizations of (1) may be written

$$\left\{ \begin{array}{l} G_m F_m y = [\lambda - k(m)]\, y, \\ F_{m+1} G_{m+1} y = [\lambda - k(m+1)]\, y. \end{array} \right. \tag{2}$$

We now show that the condition for this 2-way factorization is *Condition A.*

$$\left[\frac{f'(x,m)+f'(x,m-1)}{f(x,m)-f(x,m-1)}\right]' = -\,[f(x,m)-f(x,m-1)]$$

where a dash indicates differentiation with respect to x, and that if this condition is satisfied we find

$$g(x,m) = -\tfrac{1}{2}\,\frac{f'(x,m)+f'(x,m-1)}{f(x,m)-f(x,m-1)}, \tag{3}$$

$$k(m) = \tfrac{1}{4}\left[\frac{f'(x,m)+f'(x,m-1)}{f(x,m)-f(x,m-1)}\right]^2 - \tfrac{1}{2}[f(x,m)+f(x,m-1)]. \tag{4}$$

In fact identifying the two assumed factorizations of (1) with (1) itself we have

$$\begin{cases} g'(x,m)-g^2(x,m)-k(m) = f(x,m), \\ g'(x,m+1)+g^2(x,m+1)+k(m+1) = -f(x,m), \end{cases}$$

which may be written

$$\begin{cases} g'(x,m)-g^2(x,m)-k(m) = f(x,m), \\ g'(x,m)+g^2(x,m)+k(m) = -f(x,m-1), \end{cases}$$

or, adding and subtracting,

$$\begin{cases} g'(x,m) = \tfrac{1}{2}[f(x,m)-f(x,m-1)], \\ g^2(x,m)+k(m) = -\tfrac{1}{2}[f(x,m)+f(x,m-1)]. \end{cases}$$

Differentiating the second of the last two equations to eliminate the unknown $k(m)$ and using the first, the result gives the unkown $g(x,m)$ as in (3). Putting (3) into these two equations we find Condit on A and (4) respectively. That the right-hand side of (4) is constant may be seen by differentiating it and taking account of Condition A.

We now introduce the usual scalar product symbol $(\varphi,\psi) = \int \varphi\psi\, dx$ where the right-hand side is a definite integral extended over the x-interval in question. Evidently $(\varphi,\psi) = (\psi,\varphi)$. Concerning the two operators F_m, G_m we have

Theorem 1. F_m and G_m are adjoint operators, i. e. if $\varphi\psi$ vanishes at the limits of the x-interval, then $(\varphi, F_m\psi) = (G_m\varphi, \psi)$.

Proof. We have

$$(\varphi, F_m \psi) = \int \varphi F_m \psi \, dx = \int [\varphi g(x, m) \psi + \varphi \frac{d}{dx} \psi] \, dx$$

$$= \int [\psi g(x, m) \varphi - \psi \frac{d}{dx} \varphi] \, dx = \int \psi G_m \varphi \, dx$$

$$= (\psi, G_m \varphi),$$

whence the theorem is proved.

We shall denote by $y(x, \lambda, m)$ a solution of (1) corresponding to the number-pair $\{\lambda, m\}$, i.e. by (2)

$$\begin{cases} G_m \; F_m \; y(x, \lambda, m) = [\lambda - k(m)] y(x, \lambda, m), \\ F_{m+1} G_{m+1} y(x, \lambda, m) = [\lambda - k(m+1)] y(x, \lambda, m). \end{cases} \quad (5)$$

Theorem 2. If $y(x, \lambda, m)$ *is a solution of (1) corresponding to the number-pair* $\{\lambda, m\}$, *then*

$$y(x, \lambda, m-1) = F_m y(x, \lambda, m), \qquad y(x, \lambda, m+1) = G_{m+1} y(x, \lambda, m)$$

are solutions of (1) corresponding to the respective number-pairs $\{\lambda, m-1\}$, $\{\lambda, m+1\}$.

Proof. Writing (5) in the form

$$\begin{cases} G_m \; y(x, \lambda, m-1) = [\lambda - k(m)] y(x, \lambda, m) \\ F_{m+1} y(x, \lambda, m+1) = [\lambda - k(m+1)] y(x, \lambda, m) \end{cases}$$

and operating these two equations by F_m, G_{m+1} respectively, we obtain

$$\begin{cases} F_m \; G_m \; y(x, \lambda, m-1) = [\lambda - k(m)] y(x, \lambda, m-1) \\ G_{m+1} F_{m+1} y(x, \lambda, m+1) = [\lambda - k(m+1)] y(x, \lambda, m+1) \end{cases}$$

which, when compared with (5), prove the theorem.

Applying Theorem 2 with $y(x, \lambda, m-1)$ or $y(x, \lambda, m+1)$ taking the role of $y(x, \lambda, m)$, we obtain in each instance two solutions of (1), of which one is again $y(x, \lambda, m)$ except for a factor. Hence

Corollary. If we know a solution of (1) corresponding to the number-pair $\{\lambda, m\}$, *we know also a ladder of solutions of (1) corresponding to the number-pairs*

$$\cdots, \{\lambda, m-2\}, \{\lambda, m-1\}, \{\lambda, m\}, \{\lambda, m+1\}, \{\lambda, m+2\}, \cdots$$

3. The Boundary Hypothesis.

Consider first the case where the x-interval is finite. We shall assume:

Condition B. In the neighborhood of a limit α of the x-interval, $f(x, m)$ has a development of the form

$$\frac{a(m)}{(x-\alpha)^{\nu}}+\frac{a'(m)}{(x-\alpha)^{\nu-1}}+\ldots \quad (\nu>1)$$

where $a(m)$ is such that $|a(m)|>|a(m-1)|$.

Then, by (3), $g(x, m)$ has in the neighborhood of α a development of the form

$$\frac{b(m)}{x-\alpha}+\text{ a function of } x, m \text{ regular at } x=\alpha, \tag{6}$$

where

$$b(m)=\frac{\nu}{2}\,\frac{a(m)+a(m-1)}{a(m)-a(m-1)}>0.$$

Theorem 3. If $y(x, \lambda, m)$ is an eigensolution of (1), the solutions $y(x, \lambda, m-1)$ and $y(x, \lambda, m+1)$ of (1) as defined in Theorem 2 are also eigensolutions.

Proof. Since $y(x, \lambda, m)$ as eigensolution of (1) satisfies (1) and vanishes at $x=\alpha$, it must vanish at $x=\alpha$ at least as rapidly as $(x-\alpha)^{\nu}$. Hence

$$y(x,\lambda,m-1)=F_m\, y(x,\lambda,m)=g(x,m)y(x,\lambda,m)+\frac{d}{dx}y(x,\lambda,m)$$

$$y(x,\lambda,m+1)=G_{m+1}y(x,\lambda,m)=g(x,m+1)y(x,\lambda,m)-\frac{d}{dx}y(x,\lambda,m)$$

also vanish at $x=\alpha$. As these are already solutions of (1) by Theorem 2, they are therefore eigensolutions of (1).

Theorem 4. A necessary condition that there exist an eigensolution of (1) corresponding to the number-pair $\{\lambda, m\}$ is $\lambda-k(m+1)\geqq 0$.

Proof. Writing the second equation of (5) in the form

$$F_{m+1}\, y\,(x,\lambda,m+1)=[\lambda-k(m+1]\, y\,(x,\lambda,m),$$

multiplying this by $y\,(x,\lambda,m)$, and integrating the result over the x-interval, we have

$$(y(x,\lambda,m),\, F_{m+1}y(x,\lambda,m+1))=[\lambda-k(m+1)]\\ (y(x,\lambda,m),\, y(x,\lambda,m)).$$

Since the left-hand side of this relation is equal to $(G_{m+1}y(x,\lambda,m),\, y(x,\lambda,m+1))$ by Theorem 1, the relation may be written

$$(y(x,\lambda,m+1),\, y(x,\lambda,m+1))=[\lambda-k(m+1)]\\ (y(x,\lambda,m),\, y(x,\lambda,m)), \tag{7}$$

whence Theorem 4 follows.

THE K-HYPOTHESIS

We shall further assume

Condition C. $k\,(m+1)$ is an increasing function of m for $m>m_o$.

We then prove

Theorem 5. $\lambda_n=k\,(n+1)$ $(n=m_o,\, m_o+1,\, m+_o2,\, \cdots)$ are and are the only eigenvalues of (1) lying in the λ-interval $[-\infty,\, k\,(+\infty)]$.

Proof. a) We shall first show that each λ_n, which evidently lies in the interval $[-\infty,\, k\,(+\infty)]$, is an eigenvalue of (1). For this purpose it suffices to show that the second equation of (5) for $\lambda=\lambda_n$:

$$F_{m+1}\, G_{m+1}\, y(x,\lambda_n,m)=[\lambda_n-k(m+1)]\, y(x,\lambda_n,m)$$

has at least an eigensolution $y\,(x,\lambda_n,m)$ for some m. Choosing $m=n$ this equation reduces to

$$F_{n+1}\, G_{n+1}\, y\,(x,\lambda_n,n)=0.$$

If the last equation has an eigensolution, it must satisfy

$$G_{n+1}\, y\,(x,\lambda_n,n)=0. \tag{8}$$

otherwise we would have $G_{n+1}\, y\,(x,\lambda_n,n)\neq 0$ and by Theorems 2, 3,

$y(x, \lambda_n, n+1) = G_{n+1}\, y(x, \lambda_n, n)$ would be an eigensolution of (1) corresponding to the number-pair $\{\lambda_n, n+1\}$, which contradicts Theorem 4 since $\lambda_n - k(n+2) = k(n+1) - k(n+2) < 0$ (by Condition C). Now equation (8) has surely one and only one independent eigensolution: for (8) may be written

$$g(x, n+1)\, y(x, \lambda_n, n) - \frac{d}{dx} y(x, \lambda_n, n) = 0,$$

whence

$$y(x, \lambda_n, n) = C_n\, e^{\int g(x, n+1)\, dx} \tag{9}$$

where C_n is an arbitrary constant. Since in the neighborhood of a limit α of the x-interval $g(x, n+1)$ has by (6) a term of the form $b(n+1)/(x-a)$ where $b(n+1) > 0$, the right-hand side of (9) has a factor of the form $(x-a)^{b(n+1)}$ and so vanishes at $x=a$ and is therefore an eigensolution.

We have thus proved that, each λ_n is an eigenvalue of (1), there existing exactly one independent eigensolution of (1) corresponding to the number-pair $\{\lambda_n, m=n\}$, namely that given by (9).

No eigensolution of (1) can exist corresponding to a number-pair of the form $\{\lambda_n, m>n\}$, otherwise we would have $\lambda_n - k(m+1) = k(n+1) - k(m+1) < 0$ (by Condition C) contrary to Theorem 4. Hence (9) is the *top* of a ladder of eigensolutions:

$$\begin{cases} y(x, \lambda_n, n) = C_n\, e^{\int g(x, n+1)\, dx} \\ y(x, \lambda_n, n-1) = F_n\, y(x, \lambda_n, n) \\ y(x, \lambda_n, n-2) = F_{n-1}\, y(x, \lambda_n, n-1) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ y(x, \lambda_n, m_0) = F_{m_0+1}\, y(x, \lambda_n, m_0+1). \end{cases} \tag{10}$$

b) We shall next show that the λ_n are the only eigenvalues of (1) lying in the interval $[-\infty, k(+\infty)]$. Let us suppose that this is not the case so that there exists an eigenvalue λ' of (1) lying in the same interval but different from all the λ_n. We certainly have $\lambda' > k(m_0+1)$ since $\lambda' < k(m_0+1)$ would imply $\lambda' - k(m+1) < k(m_0+1) - k(m+1) \leqq 0$ (by Condition C) contrary to Theorem 4.

Hence λ', which was supposed to lie in the interval $[-\infty, k(+\infty)]$, lies in fact in the subinterval $[k(m_0+1), k(+\infty)]$. But the values in this subinterval are those of $k(m+1)$ for all $m > m_0$, whence there exists an $n' > m_0$ for which $\lambda' = k(n'+1)$ and which is not an integer. Denote by $q, q+1$ the two consecutive integers such that $q < n' < q+1$ and consider the ladder of eigensolutions corresponding to the number-pairs

$$\{\lambda', m_0\}, \{\lambda', m_0+1\}, \{\lambda', m+2\}, \cdots, \{\lambda', q\}, \{\lambda', q+1\}, \cdots.$$

The member of this ladder corresponding to $\{\lambda', q+1\}$ is certainly not identically zero, as can be seen from (7) by putting $\lambda = \lambda'$, $m = q$ and by noting that $\lambda' - k(q+1) = k(n'+1) - k(q+1) > 0$ (Condition C). But the existance of this eigensolution corresponding to the number-pair $\{\lambda', q+1\}$ is contradictory to Theorem 4 since $\lambda' - k(q+2) = k(n'+1) - k(q+2) < 0$ (by Condition C). Theorem 5 is thus completely proved.

To sum up, *the eigenvalues of* (1) *lying in the interval* $[-\infty, k(+\infty)]$ *are discrete, being of the form*

$$\lambda_n = k(n+1), \quad (n = m_0, m_0+1, m_0+2, \cdots).$$

and to each such λ_n *there corresponds a ladder of eigensolutions given by* (10).

5. NORMALIZATION

The first y in (10) will be normalized:

$$(y(x, \lambda_n, n), y(x, \lambda_n, n)) = 1$$

if we take

$$C_n = \left(e^{\int g(x,n+1)dx}, e^{\int g(x,n+1)dx} \right)^{-\frac{1}{2}}.$$

Since a constant multiple of an eigensolution is again an eigensolution corresponding to the same eigenvalue, we multiply the remaining y's in (10) by suitable constant factors and write the ladder of eigensolutions (10) in the form

$$\left\{ \begin{array}{l} y(x,\lambda_n,n) = C_n e^{\int g(x,n+1)dx}, \quad C_n = \left(e^{\int g(x,n+1)dx}, e^{\int g(x,n+1)dx} \right)^{-\frac{1}{2}} \\ y(x,\lambda_n,m) = [k(n+1) - k(m+1)]^{-\frac{1}{2}} F_{m+1} y(x,\lambda_n,m+1), \\ \quad (m = n-1, n-2, \cdots, m_0). \end{array} \right. \quad (11)$$

Here n may take each of the values $n = m_0, m_0+1, m_0+2, \cdots$.

We now show that not only the first y but also the remaining y's in (11) are normalized. In fact we have

$$
\begin{aligned}
(y(x,\lambda_n,m), y(x,\lambda_n,m)) &= [k(n+1) - k(m+1)]^{-1}(F_{m+1}y(x,\lambda, m+1), F_{m+1}\, y\,(x,\lambda_n,m+1)) \\
&= [k(n+1) - k(m+1)]^{-1}(O_{m+1}F_{m+1}y\,(x,\lambda_n,m+1),\, y\,(x,\lambda_n,m+1)) \quad \text{(by Theorem 1)} \\
&= (y(x,\lambda_n,m+1),\, y\,(x,\lambda_n,m+1)) \quad \text{(by the first equation of (5))}
\end{aligned}
$$

whence, putting $m = n-1, n-2, \cdots, m_0$ successively, we find

$$(y(x,\lambda_n,m),\, y(x,\lambda_n,m)) = 1 \qquad (m = n-1, n-2, \cdots, m_0).$$

6. Examples.

Ex. 1. *Spherical harmonics.* The associated spherical harmonics (Legendre's associated functions) are defined as eigensolutions of the Sturm-Liouville equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dv}{d\theta}\right) + \left(\lambda - \frac{m^2}{\sin^2\theta}\right)v = 0 \quad (m = \text{integer})$$

where θ lies in the finite interval $[0, \pi]$. When we make the change of dependent variable $y = v\sin^{\frac{1}{2}}\theta$, we arrive at the canonical form

$$\frac{d^2y}{d\theta^2} + \left(\lambda + \frac{1}{4} - \frac{m^2 - \frac{1}{4}}{\sin^2\theta}\right)y = 0.$$

Negative integral values of m may be omitted since they do not lead to new eigensolutions. Hence $m = 0, 1, 2, \cdots$ and $m_0 = 0$.

We have then

$$
\begin{aligned}
f(\theta,m) &= \frac{1}{4} - \frac{m^2 - \frac{1}{4}}{\sin^2\theta}, \quad g(\theta,m) = (m - \tfrac{1}{2})\cot\theta, \quad k(m+1) \\
&= m\,(m+1)
\end{aligned}
$$

and the Conditions A, B, C are all satisfied. Hence the eigenvalues are, by Theorem ,

$$\lambda_n = n\,(n+1) \quad (n = 0, 1, 2, \cdots),$$

and the normalized eigensolutions are, by (11),

$$\left\{\begin{aligned} y(\theta,\lambda_n,n) &= \left[\frac{2^{2n+1}}{2n+1}\frac{(n!)^2}{(2n)!}\right]^{-\frac{1}{2}} \sin^{n+\frac{1}{2}}\theta, \\ y(\theta,\lambda_n,m) &= \left[(n-m)\ (n+m+1)\right]^{-\frac{1}{2}} \left\{(m+\tfrac{1}{2})\cot\theta \right.\\ &\quad \left.+\frac{d}{d\theta}\right\} y(\theta,\lambda_n,m+1) \quad (m=n-1,\ n-2,\cdots,0). \end{aligned}\right.$$

Ex. 2. *The Kepler problem in a spherical space.*[1] Consider, instead of the ordinary 3-dimensional Euclidean space, a hypersphere

$$x_1^2+x_2^2+x_3^2+x_4^2=a^2$$

of radius a in 4-dimensional Euclidean space. Representing this spherical space by the parametric equations,

$$\left\{\begin{aligned} x_1 &= a \sin\rho \sin\theta \sin\varphi \\ x_2 &= a \sin\rho \sin\theta \cos\varphi \\ x_3 &= a \sin\rho \cos\theta \\ x_4 &= a \cos\rho \end{aligned}\right. \qquad [0\leqq\rho\leqq\pi,\ 0\leqq\theta\leqq\pi,\ 0\leqq\varphi<2\pi]$$

it has the fundamental form

$$ds^2=dx_1^2+dx_2^2+dx_3^2+dx_4^2=a^2[d\rho^2+\sin^2\rho(d\theta^2+\sin^2\theta d\varphi^2)].$$

When ρ is small this form approximates to $a^2\,[d\rho^2+\rho^2(d\theta^2+\sin^2\theta d\varphi^2)]$ which is the fundamental form of the ordinary 3-dimensional Euclidean space in polar coordinates, the radial variable being $r=a\rho$.

The Laplace operator of the spherical space in question is

$$\Delta=\frac{1}{a^2\sin^2\rho}\left[\frac{\partial}{\partial\rho}\sin^2\rho\frac{\partial}{\partial\rho}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right].$$

Schrödinger's equation for a particle of mass μ is

$$\Delta\psi+\frac{2\mu}{h}(E-V)\,\psi=0$$

where V denotes the potential energy of the particle. As in the

1. E. Schrodinger, Proc. Roy. Soc. A 46, 9 (1940).

classical Kepler problem we assume that V depends on ρ alone and is a harmonic function in the spherical space, i. e. satisfies the potential equation $\Delta V=0$. This equation now reads

$$\frac{d}{d\rho}\left(\sin^2\rho\,\frac{dV}{d\rho}\right)=0$$

and yields V as a constant multiple of $\cot\rho$. Assuming further that when ρ is small V approximates to the classical Coulomb energy $-Ze^2/r$, we find

$$V=-\frac{Ze^2}{a}\cot\rho.$$

Writing ψ in Schrodinger's equation as the product of a function V of ρ alone by an ordinary spherical harmonic, say of degree m $(\geqq 0)$, v satisfies the Sturm-Liouville equation

$$\frac{d}{d\rho}\left(\sin^2\rho\,\frac{dv}{d\rho}\right)+\left(\lambda\sin^2\rho+\frac{\mu aZe^2}{h^2}\sin 2\rho-m(m+1)\right)v=0$$

where

$$\lambda=\frac{2\mu a^2}{h^2}E$$

is the eigenvalue parameter. When we make the change of dependent variable $y=v\sin\rho$, we obtain the canonical form

$$\frac{d^2y}{d\rho^2}+\left(\lambda+1+\frac{2\mu aZe^2}{h^2}\cot\rho-\frac{m(m+1)}{\sin^2\rho}\right)y=0.$$

Here $m=0, 1, 2, \ldots$ and $m_0=0$.

We have now

$$f(\rho,m)=1+\frac{2\mu aZe^2}{h^2}\cot\rho-\frac{m(m+1)}{\sin^2\rho},$$

$$g(\rho,m)=-\frac{\mu aZe^2}{mh^2}+m\cot\rho,$$

$$k(m+1)-m(m+2)-\left[\frac{\mu aZe^2}{(m+1)h^2}\right]^2,$$

and the Conditions A, B, C are all satisfied. Hence the eigenvalues are, by Theorem 5,

$$\lambda_n = n(n+2) - \left[\frac{\mu a Z e^2}{(n+1)h^2}\right]^2 \quad (n=0,\ 1,\ 2,\ \cdots),$$

and the normalized eigenfunctions are, by (11),

$$y(\rho, \lambda_n, n) = \left[\frac{2^{-2n-3}(2n+2)!(n+1)}{(p^2+1^2)(p^2+2^2)\cdots(p^2+(n+1)^2)} \frac{1-e^{-2\pi p/(n+1)}}{p}\right]^{-\frac{1}{2}}$$

$$e^{-p\rho/(n+1)} \sin^{n+1}\rho \quad \left(\rho = \frac{\mu a Z e^2}{h^2}\right)$$

$$y(\rho, \lambda_n, m) = \left[(n-m)(n+m+2)\left(1+\frac{p^2\hbar^2}{(n+1)^2(m+1)^2}\right)\right]^{-\frac{1}{2}}$$

$$\left\{-\frac{p\hbar}{m+1} + (m+1)\cot\rho + \frac{d}{d\rho}\right\} y(\rho, \lambda_n, m+1)$$

$$(m = n-1,\ n-2,\ \cdots,\ 0).$$

The energy levels of the particle are then

$$E_n = \frac{\hbar^2}{2\mu a^2}\lambda_n = \frac{n(n+2)\hbar^2}{2\mu a^2} - \frac{\mu Z^2 e^4}{2(n+1)^2\hbar^2} \quad (n=0,1,2,\cdots).$$

These constitute a discrete spectrum. Unlike the theory in the Euclidean space a continuous spectrum does not arise now. For large a, E_n approximates to Bohr's expression

$$-\frac{\mu Z^2 e^4}{2(n+1)^2\hbar^2}.$$

7. INFINITE X-INTERVAL.

Consider next the case where the x-interval is infinite, either the upper limit being $+\infty$ or the lower limit $-\infty$ or both. All the above results will still hold if only Condition B is suitably modified. We separate this case into two subcases:

I. When in the neighborhood of infinity the development of $f(x, m)$ in power series of x contains only negative powers, we assume

Condition B_1. The development is of the form

$$\frac{a_0}{x^{\nu}}+\frac{a_1(m)}{x^{\nu+1}}+\cdots \qquad (\nu>0)$$

where $a_0 \neq 0$ is independent of m and $a_1(m)$ depends on m such that $a_1(m)-a_1(m-1)$ has the same sign as $\mp a_0$ when the limit is $\pm\infty$.

Then, by (3), $g(x, m)$ has in the neighborhood of infinity a development of the form

$$\frac{\nu a_0}{a_1(m)-a_1(m-1)}+\text{negative powers of } x. \qquad (6_1)$$

Only the following two modifications of our theory are now necessary;

1. Proof of Theorem 3. Since $y(x, \lambda, m)$ as eigensolution of (1) vanishes at infinity and is such that $(y(x, \lambda, m), y(x, \lambda, m))$ is finite, its development in the neighborhood of infinity is of the form

$$\frac{c}{x^{t}}+\frac{c'}{x^{t+1}}+\cdots \qquad (t>\tfrac{1}{2})$$

whence $y(x, \lambda, m-1)$ and $y(x, \lambda, m+1)$ have developments of the same form and are therefore also eigensolutions of (1).

2. Proof that (9) is an eigensolution of (1). In the neighborhood of infinity the right-hand side of (9) has, by (6_1), a factor of the form

$$e^{\frac{\nu a_0 x}{a_1(m)-a_1(m-1)}}$$

which tends to zero exponentially by Condition B_1. Hence (9), which vanishes at infinity such that $(y(x, \lambda_\varkappa, n), y(x, \lambda_\varkappa, n))$ is finite, is an eigensolution of (1).

Ex. 3. *The Kepler problem in ordinary space.* It is well known that in this problem the radial function v satisfies the Sturm-

Liouville equation

$$\frac{d}{dr}\left(r^2\frac{dv}{dr}\right)+\left(\lambda r^2+\frac{2\mu Ze^2}{\hbar^2}r-m(m+1)\right)v=0 \quad (m=\text{integer})$$

where r lies in the interval $[0, +\infty]$ and where

$$\lambda=\frac{2\mu}{\hbar^2}E$$

is the eigenvalue parameter. When we make the change of dependent variable $y=rv$, we obtain the canonical form

$$\frac{d^2y}{dr^2}+\left(\lambda+\frac{2\mu Ze^2}{\hbar^2 r}-\frac{m(m+1)}{r^2}\right)y=0.$$

Negative values of m may be neglected since they do not lead to new eigensolutions. Thus $m=0,\ 1,\ 2,\ \ldots$ and $m_0=0$.

We have here

$$f(r,m)=\frac{2\mu Ze^2}{\hbar^2 r}-\frac{m(m+1)}{r^2}, \qquad g(r,m)=-\frac{\mu Ze^2}{m\hbar^2}+\frac{m}{r},$$

$$k(m+1)=-\frac{\mu^2Z^2e^4}{(m+1)^2\hbar^4},$$

and the Conditions A, B, B_1, C are all satisfied. Hence, by Theorem 5, the *negative* eigenvalues are

$$\lambda_n=-\frac{\mu^2Z^2e^4}{(n+1)^2\hbar^4} \quad (n=0,1,2,\cdots),$$

and by (11) the normalized eigensolutions are

$$\begin{cases} y(r,\lambda_n,n)=\dfrac{e}{(n+1)\hbar}\left[\dfrac{2\mu Ze^2}{(n+1)\hbar^2}\right]^{n+1}\left[\dfrac{\mu Z}{(2n+1)!}\right]^{\frac{1}{2}} r^{n+1}e^{-\frac{\mu Ze^2}{(n+1)\hbar^2}r} \\ y(r,\lambda_n,m)=\dfrac{(n+1)(m+1)\hbar^2}{\mu Ze^2\sqrt{(n-m)(n+m+2)}}\left\{-\dfrac{\mu Ze^2}{(m+1)\hbar^2}+\dfrac{m+1}{r}+\dfrac{d}{dr}\right\}y(r,\lambda_n,m+1) \end{cases}$$

$$(m = n-1,\ n-2,\ \cdots,\ 0).$$

The *negative* energy levels of the particle in question are

$$E_n = \frac{\hbar^2}{2\mu}\lambda_n = -\frac{\mu Z^2 e^4}{2(n+1)^2\hbar^2} \quad (n = 0,1,2,\cdots).$$

This is Bohr's known expression.

II. When in the neighborhood of infinity the development of $f(x, m)$ in power series of x contains at least a positive power, we assume

Condition B$_2$. *The development is of the form*

$$a_0 x^{\nu} + \cdots + a_p(m) x^{\nu - p} + \cdots \qquad (\nu > \tfrac{1}{2})$$

where $a_0 \neq 0$, $a_p(m)$ *is the first coefficient dependent on* m, p *is a positive integer* $< \nu + \frac{1}{2}$, *and* $a_p(m) - a_p(m-1)$ *has the same sign as* $(\pm 1)^p a_0$ *when the limit is* $\pm\infty$.

Then, by (3), $g(x, m)$ has in the neighborhood of infinity a development of the form

$$-\frac{\nu a_0 x^{p-1}}{a_p(m) - a_p(m-1)} + \text{lower powers of } x.$$

The two modifications in the proofs are:

1. Proof of Theorem 3. As before, the development of the eigensolution $y(x, \lambda, m)$ of (1) in the neighborhood of infinity is of the form

$$\frac{c}{x^t} + \frac{c'}{x^{t+1}} + \cdots \qquad (t > \tfrac{1}{2}).$$

Since this $y(x, \lambda, m)$ satisfies (1), it vanishes at infinity at least as rapidly as $1/x^{\nu}$ so that we also have $t \geqq \nu$. Hence $y(x, \lambda, m-1)$, $y(x, \lambda, m+1)$ have developments of the form

$$\frac{d}{x^{t-p+1}} + \frac{d'}{x^{t-p+2}} + \cdots$$

and are eigensolutions of (1), since $t - p + 1 \geqq \nu - p + 1 > \nu - (\nu + \frac{1}{2}) + 1 = \frac{1}{2}$.

2. Proof that (9) is an eigensolution of (1). In the neighborhood of infinity the right-hand side of (9) has, by (6_2), a factor of the form

$$e^{-\frac{y}{p}\frac{a_0 x^p}{a_p(m)-a_p(m-1)}}$$

which tends to zero exponentially by Condition B_1. Hence (9), which vanishes at infinity such that $(y(x,\lambda_n,m), y(x,\lambda_n,n))$ is finite, is an eigensolution of (1).

Ex. 4. *The harmonic oscillator.* The equation for the harmonic oscillator has the form

$$\frac{d^2y}{dx^2}+(\lambda-x^2)y=0$$

where x lies in the interval $(-\infty,+\infty)$. Following Infeld we introduce an auxiliary integer m by putting $\lambda=\lambda'+2m$ so that we have

$$\frac{d^2y}{dx^2}+(\lambda'-2m-x^2)y=0$$

where λ' is regarded as eigenvalue parameter. Here $m_1=-\infty,\ldots,+\infty$, and $m_0=-\infty$.

We have now

$$f(x,m)=-x^2-2m,\ g(x,m)=-x,\ k(m+1)=2m+1,$$

and the Conditions A, B_2, C are all satisfied. Hence the eigenvalues are, by Theorem 5,

$$\lambda'_n=2n+1 \qquad (n=-\infty,\cdots,+\infty),$$

and the normalized eigensolutions are, by (11),

$$\begin{cases} y(x,\lambda'_n,n)=\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}} \\ y(x,\lambda'_n,m)=[z(n-m)]^{-\frac{1}{2}}\left\{-x+\frac{d}{dx}\right\}y(x,\lambda_n,m+1) \end{cases}$$

$$(m=n-1,\ n-2,\ \ldots,\ -\infty).$$

These functions are easily brought into relation with the Hermite polynomials.

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1) L. Infeld, *Physical Review* 59 (1941), 737.

2) E. Schrodinger, *Proc. Roy. Irish Acad.* A 46 (1940) 9.