第 59 卷 第 10 期 2010 年 10 月
1000-3290/2010/59(10)/7149-09

物 理 学 报
ACTA PHYSICA SINICA

Vol. 59, No. 10, October 2010

三层流体系统非线性界面内波传播理论的研究*

温文媄[1)]　陈小刚[1)†]　宋金宝[2)]

1)(内蒙古工业大学理学院, 呼和浩特　010051)

2)(中国科学院海洋研究所, 青岛　266071)

(2009 年 11 月 6 日收到;2010 年 1 月 18 日收到修改稿)

基于波陡很小的假设, 利用摄动法, 讨论了任意深度的二维不可压缩、无黏性、无旋的三层流体系统. 在刚性上边界、平底不可渗透条件下, 给出了界面内波传播的统一理论以及描述其波剖面的近似非线性演化方程(NEEs). 最后讨论了几种特殊情形下的近似 NEEs. 结果表明文献导出的理论结果为本文的特殊情形.

关键词: 非线性演化方程, 摄动法, 三层流体系统, 统一理论

PACC: 4735, 0340G

1. 引　　言

界面内波在海洋、湖泊中并不少见,对其进行研究不仅对与其自身相关的现象有重要意义,而且对了解界面内波在海洋内部混合以及在环境和气候变化中的作用等具有指导意义. 近年来, 人们对有限振幅内波做了广泛的研究. 然而, 大部分都是对两层分层流体系统界面内波的研究, 对多层分层流体系统界面内波的研究非常有限. 虽然两层分层流体系统非常简单, 但对于波长大于密度曲线的特征尺度的连续系统, 它能给出很好的近似. 例如, 对上边界为刚性的两层流体系统, Djordjevic 和 Redekopp[1]给出了在弱非线性长波假设下的波剖面演化方程——KdV 方程; Benjamin[2] 和 Ono[3] 考虑了下层流体无限深度时的情形, 导出了 Benjamin-Ono (BO)方程; 另外, 基于同种结构流体系统, Segur 和 Hammack[4] 研究了下层很薄而上层较厚(相对特征波长)的两层流体系统, 得到了所谓的有限深度方程[5], 这类方程将 KdV 方程和 BO 方程联系起来; 在假设两层流体厚度相对特征波长均很小时, Kakutani 和 Yamasaki[6] 证明了在快速模态情况下的自由表面及界面波剖面是由相同类型的 KdV 方程控制; Choi[7] 在某一层流体厚度与特征波长相比是一小量的假设下,给出了两层流体系统中完全非线性内波的一般演化方程. 上述的所有理论都或多或少地依赖于长波、浅水波或小振幅波的假设, 具有一定的局限性. 1993 年 Matsuno[8] 提出了一种统一理论, 这种理论是基于任意深度参数的两层流体系统的内波传播理论. 然而, 真实海洋密度成层现象非常复杂, 有时明显呈多层成层状态,利用两层界面内波理论常常难以合理地描述海洋内部的波动规律. 因此,开展多层密度成层流体界面内波统一理论的研究是非常必要的.

多层密度成层流体界面内波运动是通过非线性波动方程来描述的, 寻求其演化理论在非线性问题研究中占有非常重要的地位. 目前, 已有许多研究方法, 如摄动法[9—12]、齐次平衡法[13,14]、Jocobi 椭圆函数展开法[15,16]、双曲函数展开法[17]、正弦-余弦法[18,19] 和函数变换法[20] 等.

本文基于波陡很小的假设, 利用摄动法, 讨论任意深度的二维无黏性不可压缩、无旋的三层流体系统, 所有研究均不依赖于长波、浅水波或小振幅波的假设.

2. 基本方程和边界条件

考虑密度呈三层成层的不可混溶流体系统, 设

* 内蒙古人才基金(批准号: 2009)、内蒙古高等学校科学研究基金(批准号: NJ09066)、中国科学院海洋环流与波动重点实验室开放研究基金(批准号: KLOCAW0805)、中国科学院知识创新工程重大项目(批准号: KZCX1-YW-12)和国家自然科学基金青年科学基金(批准号: 40806008)资助的课题.

† 通讯联系人. E-mail: xiaogang_chen@ imut. edu. cn

流体无黏性、不可压缩、无旋，且忽略表面张力的影响，取流体系统的坐标系如图 1 所示.

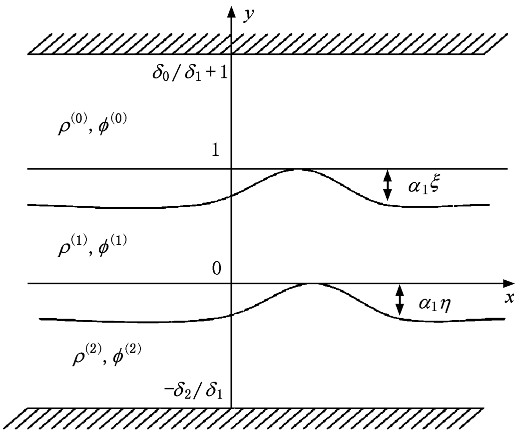

图 1 三层流体系统及界面内波示意图

连续性方程为

$$\delta_1^2\frac{\partial^2\phi^{(0)}}{\partial x^2}+\frac{\partial^2\phi^{(0)}}{\partial y^2}=0\left(1+\alpha_1\xi<y<\frac{\delta_0}{\delta_1}+1\right),\tag{1}$$

$$\delta_1^2\frac{\partial^2\phi^{(1)}}{\partial x^2}+\frac{\partial^2\phi^{(1)}}{\partial y^2}=0\ (\alpha_1\eta<y<1+\alpha_1\xi),\tag{2}$$

$$\delta_1^2\frac{\partial^2\phi^{(2)}}{\partial x^2}+\frac{\partial^2\phi^{(2)}}{\partial y^2}=0\left(-\frac{\delta_2}{\delta_1}<y<\alpha_1\eta\right).\tag{3}$$

各层流体界面上的运动学边界条件为

$$\frac{k}{\delta_1}\frac{\partial\phi^{(0)}}{\partial y}=\frac{\partial\xi}{\partial t}+k\varepsilon\frac{\partial\phi^{(0)}}{\partial x}\frac{\partial\xi}{\partial x}\quad(y=1+\alpha_1\xi),\tag{4}$$

$$\frac{k}{\delta_1}\frac{\partial\phi^{(1)}}{\partial y}=\frac{\partial\xi}{\partial t}+k\varepsilon\frac{\partial\phi^{(1)}}{\partial x}\frac{\partial\xi}{\partial x}\quad(y=1+\alpha_1\xi),\tag{5}$$

$$\frac{k}{\delta_1}\frac{\partial\phi^{(1)}}{\partial y}=\frac{\partial\eta}{\partial t}+k\varepsilon\frac{\partial\phi^{(1)}}{\partial x}\frac{\partial\eta}{\partial x}\quad(y=\alpha_1\eta),\tag{6}$$

$$\frac{k}{\delta_1}\frac{\partial\phi^{(2)}}{\partial y}=\frac{\partial\eta}{\partial t}+k\varepsilon\frac{\partial\phi^{(2)}}{\partial x}\frac{\partial\eta}{\partial x}\quad(y=\alpha_1\eta).\tag{7}$$

流体界面上的动力学边界条件为

$$\begin{aligned}&\Delta_1\left[\frac{\partial\phi^{(0)}}{\partial t}+\frac{k\varepsilon}{2\delta_1^2}\left\{\delta_1^2\left(\frac{\partial\phi^{(0)}}{\partial x}\right)^2+\left(\frac{\partial\phi^{(0)}}{\partial y}\right)^2\right\}+\xi-\xi^{(0)}\right]\\&=\frac{\partial\phi^{(1)}}{\partial t}+\frac{k\varepsilon}{2\delta_1^2}\left\{\delta_1^2\left(\frac{\partial\phi^{(1)}}{\partial x}\right)^2+\left(\frac{\partial\phi^{(1)}}{\partial y}\right)^2\right\}\\&\quad+\xi-\xi^{(0)}\quad(y=1+\alpha_1\xi),\end{aligned}\tag{8}$$

$$\begin{aligned}&\Delta\left[\frac{\partial\phi^{(1)}}{\partial t}+\frac{k\varepsilon}{2\delta_1^2}\left\{\delta_1^2\left(\frac{\partial\phi^{(1)}}{\partial x}\right)^2+\left(\frac{\partial\phi^{(1)}}{\partial y}\right)^2\right\}+\eta-\eta^{(0)}\right]\\&=\frac{\partial\phi^{(2)}}{\partial t}+\frac{k\varepsilon}{2\delta_1^2}\left\{\delta_1^2\left(\frac{\partial\phi^{(2)}}{\partial x}\right)^2+\left(\frac{\partial\phi^{(2)}}{\partial y}\right)^2\right\}\\&\quad+\eta-\eta^{(0)}\quad(y=\alpha_1\eta).\end{aligned}\tag{9}$$

上表面及下底面均为刚性边界

$$\frac{\partial\phi^{(0)}}{\partial y}=0\quad\left(y=\frac{\delta_0}{\delta_1}+1\right),\tag{10}$$

$$\frac{\partial\phi^{(2)}}{\partial y}=0\quad\left(y=-\frac{\delta_2}{\delta_1}\right).\tag{11}$$

其中 $\phi^{(i)}=\phi^{(i)}(x,y,t)\,(i=0,1,2)$ 为第 i 层流体速度势函数，$\xi=\xi(x,t)$，$\eta=\eta(x,t)$ 分别为上下两层流体界面的波面位移，$\xi^{(0)}$ 和 $\eta^{(0)}$ 是未受扰动时的界面坐标(常数)，$\Delta_1=\rho_0/\rho_1$，$\Delta=\rho_1/\rho_2$ 分别为自上而下两层流体的密度比，k 表示波数.

引入如下无因次量

$$\begin{aligned}\tilde{x}&=lx,\\\tilde{y}&=h_1y,\\\tilde{t}&=\left(\frac{l}{c_0}\right)t,\\\tilde{\phi}^{(i)}&=\left(\frac{gla}{c_0}\right)\phi^{(i)}\quad(i=0,1,2),\\(\tilde{\xi},\tilde{\eta})&=a(\xi,\eta).\end{aligned}\tag{12}$$

无因次化参数 $\varepsilon,\alpha_j,\delta_j$ 分别定义为

$$\begin{aligned}\varepsilon&=\frac{a}{l},\\\alpha_j&=\frac{a}{h_j},\\\delta_j&=\frac{h_j}{l}\quad(j=0,1,2),\end{aligned}\tag{13}$$

其中，l，a 和 c_0 分别是特征波长、振幅及波速，h_0，h_1 和 h_2 分别为三层流体的厚度.

$$\begin{aligned}c_0&=\sqrt{\frac{gl}{k}},\\k&=k(\Delta_1,\Delta,\delta_j),\\\varepsilon&=\alpha_j\delta_j\quad(j=0,1,2).\end{aligned}\tag{14}$$

其中，g 为重力加速度.

3. 几种近似非线性演化方程(NEEs)的导出

假设方程(1)—(3)满足底部条件(10)和(11)式的解形式为

$$\phi^{(0)} = -\mathrm{i}[f_0^+(x+\mathrm{i}\delta_1(y-1),t) - f_0^-(x-\mathrm{i}\delta_1(y-1),t)], \tag{15}$$

$$\phi^{(1)} = -\mathrm{i}[f_1^+(x+\mathrm{i}\delta_1 y,t) - f_1^-(x-\mathrm{i}\delta_1 y,t)] - \mathrm{i}[f^+(x-\mathrm{i}\delta_1(y-1),t) - f^-(x+\mathrm{i}\delta_1(y-1),t)], \tag{16}$$

$$\phi^{(2)} = -\mathrm{i}[f_2^+(x-\mathrm{i}\delta_1 y,t) - f_2^-(x+\mathrm{i}\delta_1 y,t)], \tag{17}$$

这里 $f_j^+(z,t)$，$f_j^-(z,t)$，$f(z,t)$ 是关于 z 的解析函数，$0<\mathrm{Im}z<2\delta_j(-2\delta_j<\mathrm{Im}z<0)$，可用以下积分表示[8,21]：

$$f_j^{\pm}(z,t) = \pm\frac{1}{4\mathrm{i}\delta_j}\int_{-\infty}^{+\infty}\coth[\pi(y-z)/2\delta_j] \times f_j(y,t)\,\mathrm{d}y \quad (j=0,1,2), \tag{18a}$$

$$f^{\pm}(z,t) = \pm\frac{1}{4\mathrm{i}\delta_1}\int_{-\infty}^{+\infty}\coth[\pi(y-z)/2\delta_1] \times f(y,t)\,\mathrm{d}y, \tag{18b}$$

其中 f_j 和 f 为任意实函数.

当 $\mathrm{Im}z\to\pm 0$ 时，$f_j^{\pm}(z,t)$ 的边界值为

$$f_j^{\pm}(x\pm\mathrm{i}0,t) = \frac{1}{2}(1\mp\mathrm{i}T_j)f_j(x,t) \quad (j=0,1,2), \tag{19a}$$

$$f^{\pm}(x\pm\mathrm{i}0,t) = \frac{1}{2}(1\mp\mathrm{i}T_1)f(x,t), \tag{19b}$$

其中 $\boldsymbol{T}_j$ 为如下奇异积分算子：

$$\boldsymbol{T}_j f(x,t) = \frac{1}{2\delta_j}P\int_{-\infty}^{+\infty}\coth[\pi(y-x)/2\delta_j] \times f(y,t)\,\mathrm{d}y \quad (j=0,1,2), \tag{20}$$

这里 P 表示 Cauchy 主值积分.

由(19a)和(20)式可知

$$f_j^+(x+\mathrm{i}0,t) - f_j^-(x-\mathrm{i}0,t) = -\mathrm{i}\boldsymbol{T}_j f_j(x,t), \tag{21a}$$

$$f_j^+(x+\mathrm{i}0,t) + f_j^-(x-\mathrm{i}0,t) = f_j(x,t) \quad (j=0,1,2). \tag{21b}$$

类似于 Matsuno[8]，由(18a)可知

$$\begin{aligned} &f_1^+(x+\mathrm{i}\delta_1,t) - f_1^-(x-\mathrm{i}\delta_1,t)\\ &= \frac{1}{4\mathrm{i}\delta_1}\int_{-\infty}^{+\infty}\coth[\pi(y-x-\mathrm{i}\delta_1)/2\delta_1]f_1(y,t)\,\mathrm{d}y\\ &\quad + \frac{1}{4\mathrm{i}\delta_1}\int_{-\infty}^{+\infty}\coth[\pi(y-x+\mathrm{i}\delta_1)/2\delta_1]f_1(y,t)\,\mathrm{d}y\\ &= \frac{1}{4\mathrm{i}\delta_1}\int_{-\infty}^{+\infty}\{\coth[\pi(y-x-\mathrm{i}\delta_1)/2\delta_1]\\ &\quad + \coth[\pi(y-x+\mathrm{i}\delta_1)/2\delta_1]\}f_1(y,t)\,\mathrm{d}y, \end{aligned} \tag{21c}$$

又由于

$$\begin{aligned} &\coth[\pi(y-x-\mathrm{i}\delta_1)/2\delta_1]\\ &+ \coth[\pi(y-x+\mathrm{i}\delta_1)/2\delta_1]\\ &= \frac{2\left[\exp\left[\frac{\pi(y-x)}{\sigma_1}\right]-1\right]}{\exp[\frac{\pi(y-x)}{\sigma_1}+1]}\\ &= 2\tanh[\pi(y-x)/2\delta_1], \end{aligned}$$

将上式代入(21c)的右边，得

$$\begin{aligned} &f_1^+(x+\mathrm{i}\delta_1,t) - f_1^-(x-\mathrm{i}\delta_1,t)\\ &= \frac{1}{4\mathrm{i}\delta_1}\int_{-\infty}^{+\infty}2\tanh[\pi(y-x)/2\delta_1]f_1(y,t)\,\mathrm{d}y\\ &= \frac{1}{2\mathrm{i}\delta_1}\int_{-\infty}^{+\infty}\tanh[\pi(y-x)/2\delta_1]f_1(y,t)\,\mathrm{d}y\\ &= \frac{\mathrm{i}^2}{-2\mathrm{i}\delta_1}\int_{-\infty}^{+\infty}\tanh[\pi(y-x)/2\delta_1]f_1(y,t)\,\mathrm{d}y\\ &= \frac{\mathrm{i}}{-2\delta_1}\int_{-\infty}^{+\infty}\tanh[\pi(y-x)/2\delta_1]f_1(y,t)\,\mathrm{d}y\\ &= -\mathrm{i}L_1 f_1(x,t), \end{aligned}$$

即

$$f_1^+(x+\mathrm{i}\delta_1,t) - f_1^-(x-\mathrm{i}\delta_1,t) = -\mathrm{i}\boldsymbol{L}_1 f_1(x,t), \tag{22a}$$

这里 $\boldsymbol{L}_1$ 定义为如下积分算子：

$$\boldsymbol{L}_1 f(x,t) = \frac{1}{2\delta_1}\int_{-\infty}^{+\infty}\tanh[\pi(y-x)/2\delta_1]\,f(y,t)\,\mathrm{d}y, \tag{22b}$$

同理可得

$$f_1^+(x+\mathrm{i}\delta_1,t) + f_1^-(x-\mathrm{i}\delta_1,t) = 0. \tag{22c}$$

由(19b)和(20)式可得：

$$f^+(x+\mathrm{i}0,t) - f^-(x-\mathrm{i}0,t) = -\mathrm{i}T_1 f(x,t), \tag{23a}$$

$$f^+(x+\mathrm{i}0,t) + f^-(x-\mathrm{i}0,t) = f(x,t). \tag{23b}$$

类似于(22a)和(22c)式，由(18b)可得：

$$f^+(x+\mathrm{i}\delta_1,t) - f^-(x-\mathrm{i}\delta_1,t) = -\mathrm{i}\boldsymbol{L}_1 f(x,t), \tag{24a}$$

$$f^+(x+\mathrm{i}\delta_1,t) - f^-(x-\mathrm{i}\delta_1,t) = 0. \tag{24b}$$

利用(14)和(15)—(17)式，可以得到

$$\phi_x^{(0)}\Big|_{y=1+\alpha_1\xi} = -\mathrm{i}[f_{0,x}^+(x+\mathrm{i}\varepsilon\xi,t) - f_{0,x}^-(x-\mathrm{i}\varepsilon\xi,t)], \tag{25a}$$

$$\phi_y^{(0)}\Big|_{y=1+\alpha_1\xi} = \delta_1[f_{0,x}^+(x+\mathrm{i}\varepsilon\xi,t) + f_{0,x}^-(x-\mathrm{i}\varepsilon\xi,t)], \tag{25b}$$

$$\phi_t^{(0)}\Big|_{y=1+\alpha_1\xi} = -\mathrm{i}[f_{0,t}^+(x+\mathrm{i}\varepsilon\xi,t) - f_{0,t}^-(x-\mathrm{i}\varepsilon\xi,t)]; \tag{25c}$$

$$\phi_x^{(1)}\Big|_{y=1+\alpha_1\xi} = -\mathrm{i}[f_{1,x}^{+}(x+\mathrm{i}\delta_1+\mathrm{i}\varepsilon\xi,t) - f_{1,x}^{-}(x-\mathrm{i}\delta_1-\mathrm{i}\varepsilon\xi,t)] - \mathrm{i}[f_x^{+}(x-\mathrm{i}\varepsilon\xi,t) - f_x^{-}(x+\mathrm{i}\varepsilon\xi,t)], \tag{26a}$$

$$\phi_y^{(1)}\Big|_{y=1+\alpha_1\xi} = \delta_1[f_{1,x}^{+}(x+\mathrm{i}\delta_1+\mathrm{i}\varepsilon\xi,t) + f_{1,x}^{-}(x-\mathrm{i}\delta_1-\mathrm{i}\varepsilon\xi,t)] - \delta_1[f_x^{+}(x-\mathrm{i}\varepsilon\xi,t) + f_x^{-}(x+\mathrm{i}\varepsilon\xi,t)], \tag{26b}$$

$$\phi_t^{(1)}\Big|_{y=1+\alpha_1\xi} = -\mathrm{i}[f_{1,t}^{+}(x+\mathrm{i}\delta_1+\mathrm{i}\varepsilon\xi,t) - f_{1,t}^{-}(x-\mathrm{i}\delta_1-\mathrm{i}\varepsilon\xi,t)] - \mathrm{i}[f_t^{+}(x-\mathrm{i}\varepsilon\xi,t) - f_t^{-}(x+\mathrm{i}\varepsilon\xi,t)]; \tag{26c}$$

$$\phi_x^{(1)}\Big|_{y=\alpha_1\eta} = -\mathrm{i}[f_{1,x}^{+}(x+\mathrm{i}\varepsilon\eta,t) - f_{1,x}^{-}(x-\mathrm{i}\varepsilon\eta,t)] - \mathrm{i}[f_x^{+}(x+\mathrm{i}\delta_1-\mathrm{i}\varepsilon\eta,t) - f_x^{-}(x-\mathrm{i}\delta_1+\mathrm{i}\varepsilon\eta,t)], \tag{27a}$$

$$\phi_y^{(1)}\Big|_{y=\alpha_1\eta} = \delta_1[f_{1,x}^{+}(x+\mathrm{i}\varepsilon\eta,t) + f_{1,x}^{-}(x-\mathrm{i}\varepsilon\eta,t)] - \delta_1[f_x^{+}(x+\mathrm{i}\delta_1-\mathrm{i}\varepsilon\eta,t) + f_x^{-}(x-\mathrm{i}\delta_1+\mathrm{i}\varepsilon\eta,t)], \tag{27b}$$

$$\phi_t^{(1)}\Big|_{y=\alpha_1\eta} = -\mathrm{i}[f_{1,t}^{+}(x+\mathrm{i}\varepsilon\eta,t) - f_{1,t}^{-}(x-\mathrm{i}\varepsilon\eta,t)] - \mathrm{i}[f_t^{+}(x+\mathrm{i}\delta_1-\mathrm{i}\varepsilon\eta,t) - f_t^{-}(x-\mathrm{i}\delta_1+\mathrm{i}\varepsilon\eta,t)]; \tag{27c}$$

$$\phi_x^{(2)}\Big|_{y=\alpha_1\eta} = -\mathrm{i}[f_{2,x}^{+}(x-\mathrm{i}\varepsilon\eta,t) - f_{2,x}^{-}(x+\mathrm{i}\varepsilon\eta,t)], \tag{28a}$$

$$\phi_y^{(2)}\Big|_{y=\alpha_1\eta} = \delta_1[f_{2,x}^{+}(x-\mathrm{i}\varepsilon\eta,t) + f_{2,x}^{-}(x+\mathrm{i}\varepsilon\eta,t)], \tag{28b}$$

$$\phi_t^{(2)}\Big|_{y=\alpha_1\eta} = -\mathrm{i}[f_{2,t}^{+}(x-\mathrm{i}\varepsilon\eta,t) - f_{2,t}^{-}(x+\mathrm{i}\varepsilon\eta,t)]. \tag{28c}$$

3.1. 关于 $u_0, \xi, u_1, f_1, \eta, u_2, u$ 的 NEEs 系统

为导出基于波陡很小的假设下的近似 NEEs，将(25)—(28)式关于 ε 摄动展开，且保留到 $O(\varepsilon^2)$，结合(21)—(24)式可得到

$$\phi_x^{(0)}\Big|_{y=1+\alpha_1\xi} = -\boldsymbol{T}_0 f_{0,x} + \varepsilon\xi f_{0,xx} + O(\varepsilon^2), \tag{29a}$$

$$\phi_y^{(0)}\Big|_{y=1+\alpha_1\xi} = \delta_1[f_{0,x} + \varepsilon\xi T_0 f_{0,xx} + O(\varepsilon^2)], \tag{29b}$$

$$\phi_t^{(0)}\Big|_{y=1+\alpha_1\xi} = -\boldsymbol{T}_0 f_{0,t} + \varepsilon\xi f_{0,tx} + O(\varepsilon^2); \tag{29c}$$

$$\phi_x^{(1)}\Big|_{y=1+\alpha_1\xi} = -\boldsymbol{L}_1 f_{1,x} - \boldsymbol{T}_1 f_x - \varepsilon\xi f_{xx} + O(\varepsilon^2), \tag{30a}$$

$$\phi_y^{(1)}\Big|_{y=1+\alpha_1\xi} = \delta_1[\varepsilon\xi\boldsymbol{L}_1 f_{1,xx} - f_x + \varepsilon\xi\boldsymbol{T}_1 f_{xx} + O(\varepsilon^2)], \tag{30b}$$

$$\phi_t^{(1)}\Big|_{y=1+\alpha_1\xi} = -\boldsymbol{L}_1 f_{1,t} - \boldsymbol{T}_1 f_t - \varepsilon\xi f_{tx} + O(\varepsilon^2); \tag{30c}$$

$$\phi_x^{(1)}\Big|_{y=\alpha_1\eta} = -\boldsymbol{T}_1 f_{1,x} + \varepsilon\eta f_{1,xx} - \boldsymbol{L}_1 f_x + O(\varepsilon^2), \tag{31a}$$

$$\phi_y^{(1)}\Big|_{y=\alpha_1\eta} = \delta_1[f_{1,x} + \varepsilon\eta\boldsymbol{T}_1 f_{1,xx} + \varepsilon\eta\boldsymbol{L}_1 f_{xx} + O(\varepsilon^2)], \tag{31b}$$

$$\phi_t^{(1)}\Big|_{y=\alpha_1\eta} = -\boldsymbol{T}_1 f_{1,t} + \varepsilon\eta f_{1,tx} - \boldsymbol{L}_1 f_t + O(\varepsilon^2); \tag{31c}$$

$$\phi_x^{(2)}\Big|_{y=\alpha_1\eta} = -\boldsymbol{T}_2 f_{2,x} - \varepsilon\eta f_{2,xx} + O(\varepsilon^2), \tag{32a}$$

$$\phi_y^{(2)}\Big|_{y=\alpha_1\eta} = -\delta_1[f_{2,x} - \varepsilon\eta\boldsymbol{T}_2 f_{2,xx} + O(\varepsilon^2)], \tag{32b}$$

$$\phi_t^{(2)}\Big|_{y=\alpha_1\eta} = -\boldsymbol{T}_2 f_{2,t} - \varepsilon\eta f_{2,tx} + O(\varepsilon^2). \tag{32c}$$

引入各层流体界面上速度的水平分量

$$u_j = \phi_x^{(j)}\Big|_{y=\alpha\eta} \quad (j=1,2), \tag{33a}$$

$$u_0 = \phi_x^{(0)}\Big|_{y=1+\alpha_1\xi}, \qquad u = \phi_x^{(1)}\Big|_{y=1+\alpha_1\xi}. \tag{33b}$$

根据 u_j, u_0 和 u 的定义，由(29)—(32)式通过迭代可求得 $f_j (j=0,1,2)$ 为：

$$f_{0,x} = -\tilde{\boldsymbol{T}}_0 u_0 - \varepsilon\tilde{\boldsymbol{T}}_0(\xi\tilde{\boldsymbol{T}}_0 u_{0,x}) + O(\varepsilon^2), \tag{34a}$$

$$\phi_y^{(0)}\Big|_{y=1+\alpha_1\xi} = -\delta_1[\tilde{\boldsymbol{T}}_0 u_0 + \varepsilon\{\xi u_{0,x} + \tilde{\boldsymbol{T}}_0(\xi\tilde{\boldsymbol{T}}_0 u_{0,x})\} + O(\varepsilon^2)], \tag{34b}$$

$$\left(\phi_t^{(0)}\Big|_{y=1+\alpha_1\xi}\right)_x = u_{0,t} + \varepsilon(\xi_t\tilde{\boldsymbol{T}}_0 u_{0,x} - \xi_x\tilde{\boldsymbol{T}}_0 u_{0,t}) + O(\varepsilon^2); \tag{34c}$$

$$f_x = -\tilde{\boldsymbol{T}}_1 u - \tilde{\boldsymbol{T}}_1\boldsymbol{L}_1 f_{1,x} + \varepsilon\tilde{\boldsymbol{T}}_1(\xi\tilde{\boldsymbol{T}}_1 u_x$$

$$+\xi\tilde{\boldsymbol{T}}_1\boldsymbol{L}_1 f_{1,xx})+O(\varepsilon^2), \tag{35a}$$

$$\phi_y^{(1)}\Big|_{y=1+\alpha_1\xi}=\delta_1[\tilde{\boldsymbol{T}}_1 u+\tilde{\boldsymbol{T}}_1\boldsymbol{L}_1 f_{1,x}-\varepsilon\{\xi u_x+\tilde{\boldsymbol{T}}_1(\xi\tilde{\boldsymbol{T}}_1 u_x)+\tilde{\boldsymbol{T}}_1(\xi\tilde{\boldsymbol{T}}_1\boldsymbol{L}_1 f_{1,xx})\}+O(\varepsilon^2)], \tag{35b}$$

$$\left(\phi_t^{(1)}\Big|_{y=1+\alpha_1\xi}\right)_x=u_t+\varepsilon[-\xi_t(\tilde{\boldsymbol{T}}_1 u_x+\tilde{\boldsymbol{T}}_1\boldsymbol{L}_1 f_{1,xx})+\xi_x(\tilde{\boldsymbol{T}}_1 u_t+\tilde{\boldsymbol{T}}_1\boldsymbol{L}_1 f_{1,xt})]+O(\varepsilon^2); \tag{35c}$$

$$\boldsymbol{L}_1 f_x=-u_1-\boldsymbol{T}_1 f_{1,x}+\varepsilon\eta f_{1,xx}+O(\varepsilon^2), \tag{36a}$$

$$\phi_y^{(1)}\Big|_{y=\alpha_1\eta}=\delta_1[f_{1,x}-\varepsilon\eta u_{1,x}+O(\varepsilon^2)], \tag{36b}$$

$$\left(\phi_t^{(1)}\Big|_{y=\alpha_1\eta}\right)_x=u_{1,t}+\varepsilon(\eta_x f_{1,tx}-\eta_t f_{1,xx})+O(\varepsilon^2); \tag{36c}$$

$$f_{2,x}=-\tilde{\boldsymbol{T}}_2 u_2+\varepsilon\tilde{\boldsymbol{T}}_2(\eta\tilde{\boldsymbol{T}}_2 u_{2,x})+O(\varepsilon^2), \tag{37a}$$

$$\phi_y^{(2)}\Big|_{y=\alpha_1\eta}=-\delta_1[-\tilde{\boldsymbol{T}}_2 u_2+\varepsilon\{\eta u_{2,x}+\tilde{\boldsymbol{T}}_2(\eta\tilde{\boldsymbol{T}}_2 u_{2,x})\}+O(\varepsilon^2)], \tag{37b}$$

$$\left(\phi_t^{(2)}\Big|_{y=\alpha_1\eta}\right)_x=u_{2,t}+\varepsilon(-\eta_t\tilde{\boldsymbol{T}}_2 u_{2,x}+\eta_x\tilde{\boldsymbol{T}}_2 u_{2,t})+O(\varepsilon^2). \tag{37c}$$

式中的 $\tilde{\boldsymbol{T}}_j(j=0,1,2)$ 为 $\boldsymbol{T}_j$ 的逆算子且几乎处处有 $\boldsymbol{T}_j\tilde{\boldsymbol{T}}_j=\boldsymbol{I}$（$\boldsymbol{I}$ 为单位算子），故有

$$\tilde{\boldsymbol{T}}_j f(x,t)=-\frac{1}{2\delta_j}P\int_{-\infty}^{+\infty}\frac{f(y,t)}{\sinh[\pi(y-x)/2\delta_j]}\,\mathrm{d}y \quad (j=0,1,2). \tag{38}$$

若将(34)—(37)式代入(4)—(7)式及(8)和(9)式关于 x 的偏导数中，可得到关于 $u_0,\xi,u_1,f_1,\eta,u_2,u$ 的 NEEs 系统如下：

$$\xi_t+k\tilde{\boldsymbol{T}}_0 u_0+k\varepsilon[(\xi u_0)_x+\tilde{\boldsymbol{T}}_0(\xi\tilde{\boldsymbol{T}}_0 u_{0,x})]+O(\varepsilon^2)=0, \tag{39}$$

$$\xi_t-k(\tilde{\boldsymbol{T}}_1 u+\tilde{\boldsymbol{T}}_1\boldsymbol{L}_1 f_{1,x})+k\varepsilon[(\xi u)_x+\tilde{\boldsymbol{T}}(\xi\tilde{\boldsymbol{T}}u_x)+\tilde{\boldsymbol{T}}(\xi\tilde{\boldsymbol{T}}\boldsymbol{L}_1 f_{1,xx})]+O(\varepsilon^2)=0, \tag{40}$$

$$\eta_t-kf_{1,x}+k\varepsilon(u_1\eta)_x+O(\varepsilon^2)=0, \tag{41}$$

$$\eta_t-k\tilde{\boldsymbol{T}}_2 u_2+k\varepsilon[(\eta u_2)_x+\tilde{\boldsymbol{T}}_2(\eta\tilde{\boldsymbol{T}}_2 u_{2,x})]+O(\varepsilon^2)=0, \tag{42}$$

$$\Delta_1\left[u_{0,t}+\varepsilon(\xi_t\tilde{\boldsymbol{T}}_0 u_{0,x}-\xi_x\tilde{\boldsymbol{T}}_0 u_{0,t})+\frac{k\varepsilon}{2}\{u_0^2+(\tilde{\boldsymbol{T}}_0 u_0)^2\}_x+\xi_x\right]+O(\varepsilon^2)=u_t+\xi_x+\varepsilon[kuu_x+\xi_x(\tilde{\boldsymbol{T}}_1 u_t+\tilde{\boldsymbol{T}}_1\boldsymbol{L}_1 f_{1,xt})]+O(\varepsilon^2), \tag{43}$$

$$\Delta\left[u_{1,t}+\varepsilon(\eta_x f_{1,xt}-\eta_t f_{1,xx})+\frac{k\varepsilon}{2}\{u_1^2+f_{1,x}^2\}_x+\eta_x\right]+O(\varepsilon^2)=u_{2,t}+\varepsilon(\eta_x\tilde{\boldsymbol{T}}_2 u_{2,t}-\eta_t\tilde{\boldsymbol{T}}_2 u_{2,x})+\frac{k\varepsilon}{2}\{u_2^2+(\tilde{\boldsymbol{T}}_2 u_2)^2\}_x+\eta_x+O(\varepsilon^2). \tag{44}$$

上述系统是不封闭的，为得到封闭的 NEEs 系统，通过比较(35a)和(36a)式，利用附录(A1)可得到

$$f_{1,x}=\boldsymbol{T}_1 u_1-\boldsymbol{L}_1 u+\varepsilon[-\boldsymbol{T}_1\{\eta(\boldsymbol{T}_1 u_{1,x}-\boldsymbol{L}_1 u_x)\}+\boldsymbol{L}_1\{\xi(\boldsymbol{L}_1 u_{1,x}-\boldsymbol{T}_1 u_x)\}]+O(\varepsilon^2). \tag{45}$$

这样，(39)—(45)式即为关于 $u_0,\xi,u_1,f_1,\eta,u_2,u$ 的封闭的 NEEs 系统.

3.2. 关于 u_0,ξ,u_1,η,u_2,u 的 NEEs 系统

若在上述系统中消去 $f_{1,x}$，可得到如下关于 u_0,ξ,u_1,η,u_2,u 的封闭的 NEEs 系统：

$$\xi_t+k\tilde{\boldsymbol{T}}_0 u_0+k\varepsilon[(\xi u_0)_x+\tilde{\boldsymbol{T}}_0(\xi\tilde{\boldsymbol{T}}_0 u_{0,x})]+O(\varepsilon^2)=0, \tag{46}$$

$$\xi_t-k(\boldsymbol{L}_1 u_1-\boldsymbol{T}_1 u)+k\varepsilon[(\xi u)_x+\boldsymbol{L}_1\{\eta(\boldsymbol{T}_1 u_{1,x}-\boldsymbol{L}_1 u_x)\}-\boldsymbol{T}_1\{\xi(\boldsymbol{L}_1 u_{1,x}-\boldsymbol{T}_1 u_x)\}]+O(\varepsilon^2)=0, \tag{47}$$

$$\eta_t-k(\boldsymbol{T}_1 u_1-\boldsymbol{L}_1 u)+k\varepsilon[(u_1\eta)_x+\boldsymbol{T}_1\{\eta(\boldsymbol{T}_1 u_{1,x}-\boldsymbol{L}_1 u_x)\}-\boldsymbol{L}_1\{\xi(\boldsymbol{L}_1 u_{1,x}-\boldsymbol{T}_1 u_x)\}]+O(\varepsilon^2)=0, \tag{48}$$

$$\eta_t-k\tilde{\boldsymbol{T}}_2 u_2+k\varepsilon[(\eta u_2)_x+\tilde{\boldsymbol{T}}_2(\eta\tilde{\boldsymbol{T}}_2 u_{2,x})]+O(\varepsilon^2)=0, \tag{49}$$

$$\Delta_1\left[u_{0,t}+\varepsilon(\xi_t\tilde{\boldsymbol{T}}_0 u_{0,x}-\xi_x\tilde{\boldsymbol{T}}_0 u_{0,t})+\frac{k\varepsilon}{2}\{u_0^2+(\tilde{\boldsymbol{T}}_0 u_0)^2\}_x+\xi_x\right]+O(\varepsilon^2)=u_t+\xi_x+\varepsilon[kuu_x+\xi_x(\boldsymbol{L}_1 u_{1,t}-\boldsymbol{T}_1 u_t)]+O(\varepsilon^2), \tag{50}$$

$$\Delta\left[u_{1,t}+\varepsilon\{\eta_x(\boldsymbol{T}_1 u_{1,t}-\boldsymbol{L}_1 u_t)-\eta_t(\boldsymbol{T}_1 u_{1,x}-\boldsymbol{L}_1 u_x)\}+\frac{k\varepsilon}{2}\{u_1^2+(\boldsymbol{T}_1 u_1-\boldsymbol{L}_1 u)^2+\eta_x\}_x\right]+O(\varepsilon^2)$$

$$= u_{2,t} + \varepsilon(\eta_x \tilde{\boldsymbol{T}}_2 u_{2,t} - \eta_t \tilde{\boldsymbol{T}}_2 u_{2,x}) + \frac{k\varepsilon}{2}\{u_2^2 + (\tilde{\boldsymbol{T}}_2 u_2)^2\}_x + \eta_x + O(\varepsilon^2). \quad (51)$$

3.3. 关于界面内波波剖面 ξ 和 η 的 NEEs 系统

为了导出关于 ξ 和 η 的 NEES 系统，通过(46)—(49)式，可以用 ξ,η 将 u_0,u_1,u_2,u 表示为

$$ku_0 = -\boldsymbol{T}_0\xi_t + \varepsilon[\xi\xi_{xt} + \boldsymbol{T}_0(\xi\boldsymbol{T}_0\xi_t)_x] + O(\varepsilon^2), \quad (52)$$

$$ku = -\boldsymbol{L}_1\eta_t + \boldsymbol{T}_1\xi_t + \varepsilon[\xi\xi_{xt} + \boldsymbol{T}_1\{\xi(-\boldsymbol{L}_1\eta_t + \boldsymbol{T}_1\xi_t)\}_x - \boldsymbol{L}_1\{\eta(-\boldsymbol{T}_1\eta_t + \boldsymbol{L}_1\xi_t)\}_x] + O(\varepsilon^2), \quad (53)$$

$$ku_1 = -\boldsymbol{T}_1\eta_t + \boldsymbol{L}_1\xi_t + \varepsilon[\eta\eta_{xt} + \boldsymbol{L}_1\{\xi(-\boldsymbol{L}_1\eta_t + \boldsymbol{T}_1\xi_t)\}_x - \boldsymbol{T}_1\{\eta(-\boldsymbol{T}_1\eta_t + \boldsymbol{L}_1\xi_t)\}_x] + O(\varepsilon^2), \quad (54)$$

$$ku_2 = \boldsymbol{T}_2\eta_t + \varepsilon[(\eta\eta_{xt} + \boldsymbol{T}_2(\eta\boldsymbol{T}_2\eta_t)_x] + O(\varepsilon^2). \quad (55)$$

将(52)—(55)式代入到(50)式，两边同时作用于 $\boldsymbol{R}_1\tilde{\boldsymbol{T}}_1$，且利用

$$\xi_{tt} = -k(1-\Delta_1)\boldsymbol{R}_1\tilde{\boldsymbol{T}}_1\xi_x + \boldsymbol{R}_1\tilde{\boldsymbol{T}}_1\boldsymbol{L}_1\eta_{tt} + O(\varepsilon)$$

可得

$$\begin{aligned}
&\xi_{tt} + k(1-\Delta_1)\boldsymbol{R}_1\tilde{\boldsymbol{T}}_1\xi_x - \boldsymbol{R}_1\tilde{\boldsymbol{T}}_1\boldsymbol{L}_1\eta_{tt} \\
&+ \varepsilon\boldsymbol{R}_1\tilde{\boldsymbol{T}}_1\{(1-\Delta_1)[\xi(-k(1-\Delta_1)\boldsymbol{R}_1\tilde{\boldsymbol{T}}_1\xi_x \\
&+ \boldsymbol{R}_1\tilde{\boldsymbol{T}}_1\boldsymbol{L}_1\eta_{tt})]_x + \xi_t\xi_{xt} + \boldsymbol{T}_1[\xi_t(-\boldsymbol{L}_1\eta_t \\
&+ \boldsymbol{T}_1\xi_t)]_x + \boldsymbol{T}_1[\xi(-\boldsymbol{L}_1\eta_{tt} + \boldsymbol{R}_1\boldsymbol{L}_1\eta_{tt} \\
&- k(1-\Delta_1)\boldsymbol{R}_1\xi_x)]_x - \boldsymbol{L}_1[\eta_t(-\boldsymbol{T}_1\eta_t + \boldsymbol{L}_1\xi_t)]_x \\
&- \boldsymbol{L}_1[\eta(-\boldsymbol{T}_1\eta_{tt} + \boldsymbol{R}_1\tilde{\boldsymbol{T}}_1\boldsymbol{L}_1^2\eta_{tt} \\
&- k(1-\Delta_1)\boldsymbol{R}_1\tilde{\boldsymbol{T}}_1\boldsymbol{L}_1\xi_x)]_x + (-\boldsymbol{L}_1\eta_t + \boldsymbol{T}_1\xi_t) \\
&\times(-\boldsymbol{L}_1\eta_t + \boldsymbol{T}_1\xi_t)_x - \frac{\Delta_1}{2}(\boldsymbol{T}_0\xi_t)_x^2 - \Delta_1\boldsymbol{T}_0(\xi_t\boldsymbol{T}_0\xi_t)_x \\
&- \Delta_1\boldsymbol{T}_0[\xi\boldsymbol{T}_0(-k(1-\Delta_1)\boldsymbol{R}_1\tilde{\boldsymbol{T}}_1\xi_x + \boldsymbol{R}_1\tilde{\boldsymbol{T}}_1\boldsymbol{L}_1\eta_{tt})]_x \\
&- \frac{\Delta_1}{2}(\xi_t^2)_x\} + O(\varepsilon^2) = 0. \qquad (56)
\end{aligned}$$

同样，将(52)—(55)式代入到(51)式有

$$\begin{aligned}
&\eta_{tt} + k(1-\Delta)\boldsymbol{R}\tilde{\boldsymbol{T}}_2\eta_x - \Delta\boldsymbol{R}\tilde{\boldsymbol{T}}_2\boldsymbol{L}_1\xi_{tt} \\
&+ \varepsilon\boldsymbol{R}\tilde{\boldsymbol{T}}_2\{(1-\Delta)[\eta(-k(1-\Delta)R\tilde{T}_2\eta_x \\
&+ \Delta\boldsymbol{R}\tilde{\boldsymbol{T}}_2\boldsymbol{L}_1\xi_{tt})]_x + \boldsymbol{T}_2(\eta_t\boldsymbol{T}_2\eta_t)_x \\
&+ \boldsymbol{T}_2[\eta\boldsymbol{T}_2(-k(1-\Delta)\boldsymbol{R}\tilde{\boldsymbol{T}}_2\eta_x + \Delta\boldsymbol{R}\boldsymbol{T}_2\boldsymbol{L}_1\xi_{tt})]_x \\
&+ \frac{1}{2}(\boldsymbol{T}_2\eta_t)_x^2 + \frac{1-\Delta}{2}(\eta_t^2)_x \\
&- \Delta\boldsymbol{L}_1[\xi_t(-\boldsymbol{L}_1\eta_t + \boldsymbol{T}_1\xi_t)]_x - \Delta\boldsymbol{L}_1[\xi(\boldsymbol{T}_1\xi_{tt} \\
&+ k(1-\Delta)\boldsymbol{R}\tilde{\boldsymbol{T}}_2\boldsymbol{L}_1\eta_x - \Delta\boldsymbol{R}\tilde{\boldsymbol{T}}_2\boldsymbol{L}_1^2\xi_{tt})]_x \\
&+ \Delta\boldsymbol{T}_1[\eta_t(-\boldsymbol{T}_1\eta_t + \boldsymbol{L}_1\xi_t)]_x + \Delta\boldsymbol{T}_1[\eta(\boldsymbol{L}_1\xi_{tt} \\
&+ k(1-\Delta)\boldsymbol{R}\tilde{\boldsymbol{T}}_2\boldsymbol{T}_1\eta_x - \Delta\boldsymbol{R}\tilde{\boldsymbol{T}}_2\boldsymbol{L}_1\boldsymbol{T}_1\xi_{tt})]_x \\
&- \frac{\Delta}{2}(-\boldsymbol{T}_1\eta_t + \boldsymbol{L}_1\xi_t)_x^2\} + O(\varepsilon^2) = 0, \qquad (57)
\end{aligned}$$

其中算子 $\boldsymbol{R}$ 和 $\boldsymbol{R}_1$ 定义如下：

$$\boldsymbol{R} = (1 + \Delta\boldsymbol{T}_1\tilde{\boldsymbol{T}}_2)^{-1}, \boldsymbol{R}_1 = (1 + \Delta_1\boldsymbol{T}_0\tilde{\boldsymbol{T}}_1)^{-1}.$$

则(56)和(57)式即为关于 ξ 和 η 的 NEEs 系统，其中的非线性项反映了界面内波波剖面之间的非线性相互作用及非线性相互调制. 在线性近似下，(56)和(57)式可以退化为

$$\xi_{tt} + k(1-\Delta_1)\boldsymbol{R}_1\tilde{\boldsymbol{T}}_1\xi_x - \boldsymbol{R}_1\tilde{\boldsymbol{T}}_1\boldsymbol{L}_1\eta_{tt} = 0, \quad (58)$$

$$\eta_{tt} + k(1-\Delta)\boldsymbol{R}\tilde{\boldsymbol{T}}_2\eta_x - \Delta\boldsymbol{R}\tilde{\boldsymbol{T}}_2\boldsymbol{L}_1\xi_{tt} = 0. \quad (59)$$

4. 讨 论

我们已经给出了任意深度下三层流体系统的近似 NEEs. 在一些特殊情形下，NEEs 可以得到很大的简化. 下面分别讨论.

4.1. $\boldsymbol{\rho^{(0)} = 0}$

1)用 $\xi_x = -u_t + O(\varepsilon)$ 代入(43)式，则(43)式退化成文献[8]中的(3.37)式，(40)—(44)式与上边界为自由面的两层流体系统所得的结论(见文献[8]中的(3.36)，(3.37)和(3.42)—(3.44)式)相同.

2)将 $\xi_x = -u_t + O(\varepsilon)$ 代入(51)式，则(51)式退化成文献[8]中的(3.49)式，(47)—(51)式与上边界为自由面的两层流体系统所得封闭系统(见文献[8]中的(3.46)—(3.50)式)相同.

3) $\boldsymbol{R}_1 = (1 + \Delta_1\boldsymbol{T}_0\tilde{\boldsymbol{T}}_1)^{-1}$ 退化为单位算子，则(56)式对应于文献[8]中的(3.54)式，(57)式与文献[8]中的(3.55)式形式相同.

4.2. $\boldsymbol{\delta_0 = O(1)}$, $\boldsymbol{\delta_1 \ll 1}$, $\boldsymbol{\delta_2 \ll 1}$, $\boldsymbol{\delta_2 = O(\delta_1^2)}$, $\boldsymbol{\alpha_1 = O(\delta_1)}$

此时，第一层流体介于深水与浅水之间，而下面两层流体的厚度很小(与第一层流体比较). 利用附录中(A2)，(A4)，(A5—A7)，(A10)和(A11)，结合(14)式，可将(56)式和(57)式化为

$$\xi_{tt} - \xi_{xx} - \eta_{tt} - \Delta_1\delta_1\boldsymbol{T}_0(\eta_{tt} + \xi_{xx})_x$$

$$+\alpha_1[(\eta-\xi)\xi_x+(\xi_t-\eta_t)\int_{-\infty}^{+\infty}\mathrm{sgn}(y-x)\times(\xi_t-\eta_t)\mathrm{d}y]_x+O(\alpha_1\delta_1,\delta_1^2)=0, \tag{60}$$

$$\eta_{tt}-\frac{(1-\Delta)\delta_2}{(1-\Delta_1)\delta_1}\eta_{xx}-\frac{\Delta\delta_2}{\delta_1}\xi_{tt}+\alpha_1\Big[\frac{\delta_1}{\delta_2}\eta_t\int_{-\infty}^{+\infty}\mathrm{sgn}(y-x)\eta_t\mathrm{d}y+\Delta\delta_1\boldsymbol{T}_1\Big(\eta_t\int_{-\infty}^{\infty}\mathrm{sgn}(y-x)\eta_t\mathrm{d}y\Big)_x\Big]+\frac{1-\Delta}{2(1-\Delta_1)}\eta\int_{-\infty}^{+\infty}\mathrm{sgn}(y-x)\eta_{xx}\mathrm{d}y+\frac{\Delta}{2}\eta\int_{-\infty}^{+\infty}\mathrm{sgn}(y-x)\xi_{tt}\mathrm{d}y]_x+O(\alpha_1\delta_1,\delta_1^2)=0, \tag{61}$$

其中

$$k=\frac{1}{(1-\Delta_1)\delta_1}. \tag{62}$$

4.3. $\delta_0\gg1$，$\delta_1\ll1$，$\delta_2\ll1$，$\delta_2=O(\delta_1^2)$，$\delta_1=O(\alpha_1)$

此时，界面内波的波长分别远大于下面两层流体的厚度，但比第一层流体的厚度小. 最简单的方法是在(60)式中令 $\delta_0\to\infty$，可得相应的 NEE：

$$\xi_{tt}-\xi_{xx}-\eta_{tt}-\Delta_1\delta_1\boldsymbol{H}(\eta_{tt}+\xi_{xx})_x+\alpha_1[(\eta-\xi)\xi_x+(\xi_t-\eta_t)\int_{-\infty}^{+\infty}\mathrm{sgn}(y-x)\times(\xi_t-\eta_t)\mathrm{d}y]_x+O(\alpha_1\delta_1,\delta_1^2)=0, \tag{63}$$

这里算子 $\boldsymbol{H}$ 通过以下 Hilbert 变换定义为

$$\boldsymbol{H}f(x,t)=\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{f(y,t)}{y-x}\mathrm{d}y. \tag{64}$$

4.4. $\delta_0=\delta_1=O(\delta_2)\ll1,\alpha_2=O(\delta_2^2)$

此时，上面两层流体的厚度相同，且与第三层流体厚度同阶，但相对特征波长来说为小量.

将参数 k 取作

$$k=\frac{1+\Delta_1}{(1+\Delta_1)\delta_1}, \tag{65}$$

并利用附录中(A2)，(A4)，(A8)，(A9)及(A12—A15)，可得到相应于(60)和(61)式的表达式为

$$\xi_{tt}-\xi_{xx}-\frac{\delta_1^2}{3}\xi_{xxxx}-\frac{1}{1+\Delta_1}\eta_{tt}-\frac{\delta_1^2}{2(1+\Delta_1)}\eta_{ttxx}+\frac{\alpha_2\Delta_1\delta_2}{\delta_1(1+\Delta_1)}\times\Big[\frac{1}{\Delta_1}(\xi_t-\eta_t)\int_{-\infty}^{+\infty}\mathrm{sgn}(y-x)(\xi_t-\eta_t)\mathrm{d}y+\frac{1}{2(1+\Delta_1)}(\eta-2\xi)\int_{-\infty}^{+\infty}\mathrm{sgn}(y-x)\eta_{tt}\mathrm{d}y+\frac{1}{\Delta_1}(\eta-\xi)\xi_x-\xi_t\int_{-\infty}^{+\infty}\mathrm{sgn}(y-x)\xi_t\mathrm{d}y-\frac{1}{2}\xi\int_{-\infty}^{+\infty}\mathrm{sgn}(y-x)\xi_{xx}\mathrm{d}y\Big]_x+O(\alpha_2\delta_2^2,\delta_2^4)=0, \tag{66}$$

$$\eta_{tt}-\frac{(1+\Delta_1)(1-\Delta)\delta_2}{(1-\Delta_1)(\delta_1+\Delta\delta_2)}\times\Big[\eta_{xx}+\frac{\delta_1\delta_2(\Delta\delta_1+\delta_2)}{3(\delta_1+\Delta\delta_2)}\eta_{xxxx}\Big]-\frac{\Delta\delta_2}{\delta_1+\Delta\delta_2}\Big[\frac{\delta_1(\delta_1^2+2\delta_2^2+3\Delta\delta_1\delta_2)}{6(\delta_1+\Delta\delta_2)}\xi_{ttxx}+\xi_{tt}\Big]+\frac{\alpha_2\delta_1}{(\delta_1+\Delta\delta_2)}\Big[\eta_t\int_{-\infty}^{+\infty}\mathrm{sgn}(y-x)\eta_t\mathrm{d}y-\frac{\Delta\delta_2^2}{\delta_1^2}(\xi_t-\eta_t)\int_{-\infty}^{+\infty}\mathrm{sgn}(y-x)(\xi_t-\eta_t)\mathrm{d}y+\frac{\Delta\delta_2}{2(\delta_1+\Delta\delta_2)}\eta\int_{-\infty}^{+\infty}\mathrm{sgn}(y-x)\xi_{tt}\mathrm{d}y+\frac{\Delta\delta_2^2}{2\delta_1(\delta_1+\Delta\delta_2)}(\eta-\xi)\int_{-\infty}^{+\infty}\mathrm{sgn}(y-x)\xi_{tt}\mathrm{d}y+\frac{(1+\Delta_1)(1-\Delta)\delta_2}{2(1-\Delta_1)(\delta_1+\Delta\delta_2)}\eta\int_{-\infty}^{+\infty}\mathrm{sgn}(y-x)\eta_{xx}\mathrm{d}y-\frac{\Delta(1+\Delta_1)(1-\Delta)\delta_2^3}{(1-\Delta_1)(\delta_1+\Delta\delta_2)\delta_1^2}\times\Big(\xi\eta_x+\frac{1}{2}\eta\Big(\int_{-\infty}^{+\infty}\mathrm{sgn}(y-x)\eta_x\mathrm{d}y\Big)_x\Big)\Big]_x+O(\alpha_2\delta_2^2,\delta_2^4)=0. \tag{67}$$

5. 结　　论

本文在波陡很小的假设下，利用摄动法，讨论了任意深度的二维无黏性不可压缩、无旋的三层流体系统. 在上边界为刚性、底部为平底情况下，给出了满足连续性方程和底部边界条件解析解的一般形式，研究了界面内波传播的统一理论，并导出了描述界面内波波剖面的近似 NEEs. 最后给出了几种特殊情形下相应的 NEEs. 由于该系统的导出不依赖于长波、浅水波或小振幅波的假设，因此所得理论可以广泛地应用于真实的物理问题中. 结果表明：关于 ξ 和 η 的封闭 NEEs 系统(56)和(57)式中的非线性项反映了界面内波波剖面之间的非线

性相互作用及非线性相互调制；当 $\rho^{(0)} = 0$ 时，$\boldsymbol{R}_1 = (1 + \Delta_1 \boldsymbol{T}_0 \tilde{\boldsymbol{T}}_1)^{-1}$ 退化为单位算子，本文得到的封闭 NEEs 系统可以退化为文献[8]中所给出的上边界为自由面的两层流体系统的封闭 NEEs 系统，故文献[8]中所给出的上边界为自由面的两层流体系统的结果是本文研究结果的特例.

附 录 A

下面这些算子公式可通过 Fourier 变换和 Cauchy 残数定理得证.

$$(\boldsymbol{T}_j + \tilde{\boldsymbol{T}}_j)f = \tilde{\boldsymbol{T}}_j \boldsymbol{L}_j^2 f \quad (j = 0,1) \tag{A1}$$

$$\boldsymbol{T}_j f = \frac{1}{2\delta_j}\int_{-\infty}^{+\infty} \mathrm{sgn}(y - x) f(y)\,\mathrm{d}y + \frac{\delta_j}{3} f_x + \frac{\delta_j^3}{45} f_{xxx} + O(\delta_j^5)(\delta_j \to 0,\ j = 0,1,2) \tag{A2}$$

$$\tilde{\boldsymbol{T}}_j f = -\delta_j f_x - \frac{\delta_j^3}{3} f_{xxx} + O(\delta_j^5) \quad (\delta_j \to 0) \tag{A3}$$

$$\boldsymbol{L}_1 f = \frac{1}{2\delta_1}\int_{-\infty}^{+\infty} \mathrm{sgn}(y - x) f(y)\,\mathrm{d}y - \frac{\delta_1}{6} f_x - \frac{7}{360}\delta_j^3 f_{xxx} + O(\delta_j^5) \quad (\delta_1 \to 0) \tag{A4}$$

$$\tilde{\boldsymbol{T}}_1 \boldsymbol{L}_1 f = f + \frac{\delta_1^2}{2} f_{xx} + O(\delta_1^4) \quad (\delta_1 \to 0) \tag{A5}$$

$$\boldsymbol{R} f = f + \Delta\delta_2 \boldsymbol{T}_1 f_x + O(\delta_2^2) \quad (\delta_2 \to 0) \tag{A6}$$

$$\boldsymbol{R}_1 f = f + \Delta_1\delta_1 \boldsymbol{T}_0 f_x + O(\delta_1^2) \quad (\delta_1 \to 0) \tag{A7}$$

$$\boldsymbol{R} f = \left(1 + \frac{\Delta\delta_2}{\delta_1}\right)^{-1}\left[f + \frac{\Delta\delta_2(\delta_1^2 - \delta_2^2)}{3(\delta_1 + \Delta\delta_2)} f_{xx} + O(\delta_2^4)\right] \quad (\delta_1 = O(\delta_2) \to 0) \tag{A8}$$

$$\boldsymbol{R}_1 f = \left(1 + \frac{\Delta_1\delta_1}{\delta_0}\right)^{-1}\left[f + \frac{\Delta_1\delta_1(\delta_0^2 - \delta_1^2)}{3(\delta_0 + \Delta_1\delta_1)} f_{xx} + O(\delta_1^4)\right] \quad (\delta_0 = O(\delta_1) \to 0) \tag{A9}$$

$$\boldsymbol{R}\tilde{\boldsymbol{T}}_2 f = -\delta_2 f_x - \Delta\delta_2^2 \boldsymbol{T}_1 f_{xx} + O(\delta_2^3) \quad (\delta_2 \to 0) \tag{A10}$$

$$\boldsymbol{R}_1\tilde{\boldsymbol{T}}_1 f = -\delta_1 f_x - \Delta_1\delta_1^2 \boldsymbol{T}_0 f_{xx} + O(\delta_1^3) \quad (\delta_1 \to 0) \tag{A11}$$

$$\boldsymbol{R}\tilde{\boldsymbol{T}}_2 f = -\delta_2\left(1 + \frac{\Delta\delta_2}{\delta_1}\right)^{-1}\left[f_x + \frac{\delta_1\delta_2(\Delta\delta_1 + \delta_2)}{3(\delta_1 + \Delta\delta_2)} f_{xxx} + O(\delta_2^4)\right] \quad (\delta_1 = O(\delta_2) \to 0) \tag{A12}$$

$$\boldsymbol{R}_1\tilde{\boldsymbol{T}}_1 f = -\delta_1\left(1 + \frac{\Delta_1\delta_1}{\delta_0}\right)^{-1}\left[f_x + \frac{\delta_0\delta_1(\Delta_1\delta_0 + \delta_1)}{3(\delta_0 + \Delta_1\delta_1)} f_{xxx} + O(\delta_1^4)\right] \quad (\delta_0 = O(\delta_1) \to 0) \tag{A13}$$

$$\boldsymbol{R}\tilde{\boldsymbol{T}}_2 \boldsymbol{L}_1 f = \frac{\delta_2}{\delta_1}\left(1 + \frac{\Delta\delta_2}{\delta_1}\right)^{-1}\left[f + \frac{\delta_1(\delta_1^2 + 2\delta_2^2 + 3\Delta\delta_1\delta_2)}{6(\delta_1 + \Delta\delta_2)} f_{xx}\right] + O(\delta_2^4) \quad (\delta_1 = O(\delta_2) \to 0) \tag{A14}$$

$$\boldsymbol{R}_1\tilde{\boldsymbol{T}}_1 \boldsymbol{L}_0 f = \frac{\delta_1}{\delta_0}\left(1 + \frac{\Delta_1\delta_1}{\delta_0}\right)^{-1}\left[f + \frac{\delta_0(\delta_0^2 + 2\delta_1^2 + 3\Delta_1\delta_0\delta_1)}{6(\delta_0 + \Delta_1\delta_1)} f_{xx}\right] + O(\delta_1^4) \quad (\delta_0 = O(\delta_1) \to 0). \tag{A15}$$

这里 $\boldsymbol{L}_0$ 定义为如下积分算子：

$$\boldsymbol{L}_0 f(x,t) = \frac{1}{2\delta_0}\int_{-\infty}^{+\infty} \tanh[\pi(y - x)/2\delta_0]\, f(y,t)\,\mathrm{d}y.$$

[1] Djordjevic V D, Redekopp L G 1978 *J. Phys. Oceanogr.* **8** 1016

[2] Benjamin T B 1967 *J. Fluid Mech.* **29** 559

[3] Ono H 1975 *J. Phys. Soc. Jpn.* **39** 1082

[4] Segur H, Hammack J L 1982 *J. Fluid Mech.* **118** 285

[5] Joseph R I 1977 *J. Phys. A: Math. Gen.* **10** L225

[6] Kakutani T, Yamasaki N 1978 *J. Phys. Soc. Jpn.* **45** 674

[7] Choi W 1999 *J. Fluid Mech.* **396** 1

[8] Matsuno Y 1993 *J. Phys. Soc. Jpn.* **62** 1902

[9] Chen X G, Song J B, Sun Q 2005 *Acta Phys. Sin.* **54** 5699 (in Chinese)[陈小刚、宋金宝、孙 群 2005 物理学报 **54** 5699]

[10] Hu D M 1985 *Acta Oceanol. Sin.* **7** 540 (in Chinese)[胡定明 1985 海洋学报 **7** 540]

[11] Chen X G, Song J B 2006 *Chin. Phys.* **15** 756

[12] Chen X G, Guo Z P, Song J B 2008 *Chin. Phys.* B **17** 3387

[13] Fan E G, Zhang H Q 1998 *Acta Phys. Sin.* **47** 353 (in Chinese)[范恩贵、张鸿庆 1998 物理学报 **47** 353]

[14] Fan E G 2000 *Acta Phys. Sin.* **49** 1409 (in Chinese)[范恩贵 2000 物理学报 **49** 1409]

[15] Liu S K, Fu Z T, Liu S D, Zhao Q 2001 *Acta Phys. Sin.* **50** 2068 (in Chinese)[刘式适、付遵涛、刘式达、赵 强 2001 物理学报 **50** 2068]

[16] Fu Z T, Liu S K, Liu S D 2004 *Acta Phys. Sin.* **53** 343 (in Chinese)[付遵涛、刘式适、刘式达 2004 物理学报 **53** 343]

[17] Lü K P, Shi Y R, Duan W S, Zhao J B 2001 *Acta Phys. Sin.* **50** 2074 (in Chinese)[吕克璞、石玉仁、段文山、赵金保 2001 物理学报 **50** 2074]

[18] Yan Z Y, Zhang H Q, Fan E G 1999 *Acta Phys. Sin.* **48** 1 (in Chinese)[闫振亚、张鸿庆、范恩贵 1999 物理学报 **48** 1]

[19] Yan Z Y, Zhang H Q 1999 *Acta Phys. Sin.* **48** 1962 (in Chinese)[闫振亚、张鸿庆 1999 物理学报 **48** 1962]

[20] Naranmandula, Wunenboyn, Wang K X 2004 *Acta Phys. Sin.* **53** 11(in Chinese)[那仁满都拉、乌恩宝音、王克协 2004 物理学报 **53** 11]

[21] Matsuno Y 1992 *Phys. Rev. Lett.* **69** 609

The theory of nonlinear interfacial-internal wave propagation in three-layer fluid systems*

Wen Wen-Ying[1)] Chen Xiao-Gang[1)†] Song Jin-Bao[2)]

1)(*College of Science*, *Inner Mongolia University of Technology*, *Hohhot* 010051, *China*)

2)(*Institute of Oceanology*, *Chinese Academy of Sciences*, *Qingdao* 266071, *China*)

(Received 6 November 2009; revised manuscript received 18 January 2010)

Abstract

Based on the small steepness parameter assumption, the three-layer incompressible, inviscid and irrotational fluid system of arbitrary depth is discussed by using the perturbation method, and a unified theory of nonlinear interfacial-internal wave propagation and the approximate nonlinear evolution equations (NEEs) for interfacial-internal elevations are given on the basis of the rigid upper boundary and the flat impermeable bottom. At last we also discuss on NEEs arising from various limiting cases of fluid depth. It is also noted that the theories obtained from the present work include the theoretical results derived by Yoshimasa Matsuno (1993) as special cases.

Keywords: nonlinear evolution equations(NEEs), perturbation method, three-layer fluid systems, unified theory

PACC: 4735, 0340G

* Project supported by the Talent Foundation of Inner Mongolia (Grant No. 2009), the Research Fund of Higher Education of Inner Mongolia, China (Grant No. NJ09066), the Open Fund of the Key Laboratory of Ocean Circulation and Waves, Chinese Academy of Sciences, China (Grant No. KLOCAW0805), the Knowledge Innovation Program of the Chinese Academy of Sciences, China (Grant No. KZCX1-YW-12) and the Young Scientists Fund of the National Natural Science Foundation of China (Grant No. 40806008).

† Corresponding author. E-mail: xiaogang_chen@ imut. edu. cn